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Report on geometry for primary grades, classroom helps for both lower and upper elementary grades, and official registration reports for all NCTM meetings of 1960-61.

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an official journal of The National Council of Teachers of Mathematics

Editor:
E. GLENADINE GIBB
State College of Iowa
Cedar Falls, Iowa

Associate Editor:
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As we read

E. W. HAMILTON *Associate Editor*

A countrywide survey would reveal few, if any, teachers who do not profess to be teaching meaningfully a newer version of arithmetic, a better curriculum than was offered to them in 1951 or '41 or '21. Arithmetic has been socialized, objectified, integrated, and professionalized; it has been approached incidentally and accidentally as well as systematically; it has been subjected to diagnostic and detailed research and judged by its social utility as well as its cultural contributions. It has even been the battleground for competing theories of learning. Is it any wonder then, that despite all our protestations, our practice tends to be only slightly modified; that many teachers harbor the private conviction that the way they learned it was good enough; and that, if they will just sit tight and keep still, the pendulum will swing back as it does in so many human affairs?

The thing that is likely to prevent this from happening to the present activities in the field of elementary-school mathematics is the fact that the changes taking place are in the subject matter itself rather than in the approach or in the mere updating of the applications. Thus, the children and their parents are going to be able to judge firsthand the effectiveness of the changes. Since preliminary results indicate that they find this mathematical material attractive and challenging, teachers are going to be faced with two alternatives. Either they must learn to deal with it and thus maintain their position of leadership, or they must defend themselves and the old curriculum in a rear-guard action. This decision for some of you may not be any farther away than next year's adoption of a greatly improved textbook or the contagion of some new child in your classroom or some new parent in your PTA.

We speak of new material. It is new to the elementary school and it is being newly graded and arranged by various persons and groups, but most of it is an adaptation of ideas that have already been found to be intriguing in their own right as well as important to the development of other mathematics. Some of these ideas go back 2,000 years, many of them only 100 or less. Age is not the criterion. Serviceability in leading to certain kinds of thinking and in developing attitudes is the issue and newness is newness only in the sense of a succeeding position in the curriculum and unfamiliarity to many elementary teachers.

Several of the articles in this issue deal with aspects of this new material. Seeger's "Fingerprints," in particular, reviews a number of mathematical notions in a manner simple enough to appeal to youngsters almost as presented. In contrast, his account of some of the old ideas about number and the uses to which they have been put makes our ancestors look pretty silly. How many of our present pet notions will look as foolish to our great-grandchildren?

There is a slight overlap in the discussion of binary arithmetic by both Seeger and Willerding. However, Willerding's use of it along with other systems of notation

is in support of an engaging thesis. She argues that since the teacher is obligated to do what she can for the less talented as well as for the advanced, and since there is considerable evidence that a grasp of the notational system is basic to success, the study of systems and how they work would be appropriate and more likely to hold the interest of repeaters and slow learners than a second or third repetition of the same old so-called practical applications. The point that new material must be employed is well taken and the argument for this particular material is plausible.

The summary of geometric defintions by Rutland and Hosier will serve to bring into focus ideas most teachers encountered in high-school geometry but with a somewhat more modern terminology and with added clarity regarding the elements, such as, point, line, plane, and surface.

Nulton, in presenting an extreme case to emphasize a point, seems to overdraw the little girl's frustration. But, just a week ago a college girl became ill in one of my classes during an inventory test. Just the distasteful associations upset her according to her own admission. The latter part of Nulton's article presents at some length suggestions for inspiring young children. Could these learning situations be programed for machines and, if so, could they be so managed that the group stimulus is still present and the interactions of the group still effective?

Most of us can remember favorite teachers and those not so favorite. It is common for college students to identify the beginning of disability in mathematics with some poor teacher in the grades. Guggenheim's research study on classroom climate, while not as definitive as we always hope such studies will be, nevertheless furnishes some food for thought.

Considering the care with which the matching and the statistical treatment were carried out, it seems there remain only three conclusions possible:

1 The rating instrument used on the teachers must have failed.
2 The measuring instrument used on the students must have been powerless.
3 There really was no difference.

The first of these conclusions seems hardly tenable in view of the reputation of the Wrightstone Scale. The second conclusion is just as untenable for the instrument, although not as widely known, is evidently satisfactory for some measurement purposes or it would be discarded. Besides, almost any test of appropriate difficulty would show growth, and, presumably, differential growth over a year's time at this level. That leaves the third conclusion. Some speculation as to why it is true might be in order.

First, in the light of the dominant teacher's better position in preparing students for traditional tests, the result might have been expected. Second, time lapse as suggested by the author might show some different results. There is no assurance that it would, but delayed rather than immediate recall is considered to be a better measure of efficiency in teaching. Third, and certainly interesting from the professional point of view, nothing was said about the subject-matter competence and enthusiasm of the teachers. Had they been rated on these, the outcome might also have been different.

In any case, hiring and assigning may be done on the basis of academic capabilities with some assurance that personality and management qualities are *not* all-important.

This is a heavy issue—not in pages so much as in suggestions, questions, and implications. If the attempt at editorial interpretation has added to the weight or made the ideas more controversial, that was the intention!

Fingerprints

RAYMOND J. SEEGER

National Science Foundation, Washington, D.C.

Dr. Seeger is Deputy Assistant Director for Mathematical, Physical, and Engineering Sciences. He is spending this year at Oxford University under the Foundation's Advanced Study Program.

Editor's Note

This talk was given to the Lower School (Grades 4–7) of the National Cathedral School for Girls, Washington, D.C., January, 1959.

Nowadays you can hardly pick up a newspaper without reading about digital computing machines. The very first such computing machines, of course, were really our ten fingers (the Latin word for finger is *digitus*). Every once in a while I find myself still using my fingers when I wish to see if I have enough places for the people at a party, or to know how many blocks there are from C to K Streets. Do you know the trick of multiplying numbers between 5 and 9 by the use of your fingers? Let us multiply 9 by 7. On one hand turn down four fingers, the difference between 5 and 9; on the other hand, turn down two fingers, the difference between 5 and 7. Now add all the fingers that are turned down (4 and 2), and multiply those that are not turned down on each hand (1 by 3); you get 6 and 3, respectively. The number is 63. (Those of you in the seventh grade might like to discuss with your teacher why this rule always works.)

It is not at all surprising to find fingerprints on all our arithmetic, which dates back to the Greek theory of numbers, αριθμός. The number sense is almost peculiar to man as an animal; e.g., monkeys do not have it. There are, however, some notable exceptions.[1] For example, a bird may notice that one egg out of four is missing, but not that two are. There is a story about a squire who observed a crow making a nest in a watchtower. Whenever he entered the tower, the crow would leave and not return until the squire, too, had left. If two people entered the tower and only one left, the crow would not return until the other one also had gone. The crow was apparently able to keep "count" until the number of people entering the tower was five. The solitary wasp, indeed, will always distinguish five, twelve, or twenty-four categories for individual cells of eggs; the genus ***Eumenus*** (more noticeable for the female than for the male) differentiates between five male victims and ten female ones. Despite these rare incidents, number is a distinctive sense for man; he is not only sensitive to *two* dogs or to *two* cats, but even to ***two*** animals, such as one cat and one dog. How is he able to see in a couple of lollipops something in common with a pair of stockings? How is man able thus to answer the question "how many"?

Comparing without counting

One does not necessarily have to count to compare the sizes of two groups; one has to relate merely the individuals in one group to those in the other, as in the game of musical chairs, or in the setting of a

[1] F. Cajori, *History of Elementary Mathematics* (New York: Macmillan Co., 1917). T. Dantzig, *Number, the Language of Science* (3rd ed.; New York: Macmillan Co., 1939).

dinner table. It is said that the Persian Xerxes once determined the size of his army by observing that a myriad (ten thousand, i.e., a hundred hundreds—originally from a Greek word meaning numberless) people could be contained within the walls of the fortress Doriscus. Accordingly, he would merely keep filling Doriscus with soldiers and note the number of myriads. Sometimes the size of a heap of pebbles has been used for comparison (hence our word calculate from the Latin word for pebble). In this way Madagascar armies have been counted. Have you ever heard of English tally sticks, on which each notch corresponded to a pound of sterling money? In 1834 the burning of these old wooden records caused a costly fire in Parliament. The abacus (from a Greek word for board) is a natural development of this scheme; it is still used daily and expertly in San Francisco's Chinatown. When we thus use a number according to a certain size or class of objects, we speak of it as a *cardinal* number, meaning chief, from the Latin word for hinge.

We are interested, however, not only in the question "how many?" but also in the query "which one?" You, or you, or you? For this purpose we may use a rhyme like "eenie, meeny, miny, moe" or any other set of names which are arranged in a particular order. When we use numbers as names, we speak of them as *ordinal* numbers (from the Latin word *ordinis* for order); this is our common way of counting.

Sometimes, however, we prefer not to count by ones, but by twos; say, by couples like you and me. There is a tribe which had names for only two numbers: *urspun* for one and *okosa* for two. *Urspun okosa* stood for three, *okosa okosa* for four, etc. Any amount greater than six was called a heap. Certain African tribes still speak of one, two, and then many.[2] Some scholars believe that the French word *trois*, meaning three, was originally related to the French word *tres* for very. In our modern digital computing machines, indeed, we have found it very convenient to count by twos. In this case, one may have an electric switch which can be turned on or off and which, therefore, can answer a question with on for "yes" or with off for "no." The two elements here may be written symbolically, one and zero. On this basis, we can write for one, 1; for two, 10; for three, 11; for four, 100; for five, 101; for six, 110 (each column represents at most a power of two, i.e., 2×2, 2, 2^0). Let us multiply 2 by 3, i.e., 10×11. Using the facts that $1\times 1=1$ and $1\times 0=0$, we get 110; in other words, six:

$$\begin{array}{r} 11 \\ \times 10 \\ \hline 00 \\ 11 \\ \hline 110 \end{array}$$

On some occasions, however, we prefer to count by fives—probably because we have five fingers on a hand. The modern word pentagon, from a Greek word meaning a five-sided figure, comes originally from the Sanskrit word *pentcha* for five, which is related to the Persian word *pentch* for hand. In counting a large number of things, do you ever make four strokes and then a horizontal line through them? This is just another way to count by fives! The Romans used essentially a five-system; four vertical strokes, and then V for the fifth one, signifying 5. A 10 could be regarded as essentially two V's thus: $\overset{\vee}{\wedge}$ or X. Let us multiply 8 by 7 with Roman numerals, i.e., VIII by VII.

$$\begin{array}{l} \text{VIII} \\ \text{VII} \\ \hline \text{VIII}+\text{VIII}+(\text{XXV}+\text{V}+\text{V}+\text{V}) \end{array}$$

Collecting similar terms we get six I's, six V's and two X's; we translate the sum

[2] George Gamow, *One Two Three . . . Infinity* (New York: Viking, 1947).

into Roman notation as LVI, i.e., 56. Note that the relative positions of symbols begin to be meaningful here, e.g., IV is not the same as VI. This process is evidently quite complicated; no wonder it was not used by the average civilized Roman. Usually he had specially trained slaves to do his arithmetic for him.

Ten is a basic unit

Even more common, however, is our counting by tens; in this case presumably because we have a total of ten fingers. The Egyptian hieroglyphics consisted of an inverted V for one, viz., $\wedge$; if you had ten you wrote $\hat{\wedge}$. It is interesting that all Indo-European, Semitic, and Mongolian peoples have independent words up to ten; beyond this they have compound words up to one hundred (e.g., fifteen and fifty), and eventually new words (e.g., thousand). The word eleven, which seems to be an exception, is really *eine* (one, in German) plus *lif* (ten, in Old German), and twelve is *zwei* (two) plus *lif*. The Greeks and Hebrews, on the other hand, had a different number for each letter. Let us take as an illustration again the Greek multiplication of 7 by 8; the letter for 7 is ξ, for 8 is η, for 50 it is ν, and for 6 ζ. Hence 56 is $\nu\zeta$ or $\zeta\nu$ (the order of the letters in this instance is not important inasmuch as each symbol has a distinctive meaning). The multiplication itself becomes very complicated. Our modern arithmetic notation (called Arabic because the Arabs brought it to Europe from the Asiatic Hindus) has only nine symbols together with a symbol for nothing at all (our number 2 probably came about by joining two horizontal marks =, and the number 3 from connecting three horizontal marks ≡). What is all important in this notation is a number's position. In other words, number 56 is not the same as number 65. This system of tens has great advantages, partly because it can be so easily extended to very large numbers as well as to very small numbers. It is interesting that the United States coinage (like a dime) is based upon tens, but not its system of weights and measures (like a pound and a foot)—perhaps because the former was new and isolated. One sometimes also counts by twenties, probably due to the fact that we have ten fingers and ten toes; hence we have English words such as score, threescore, fourscore, as well as the French word *vingt* for twenty and *quatre-vingt* for eighty. The Aztecs, you may recall, actually used a day with twenty hours. In all this manner of counting, we note how important is the role of the human anatomy. One wonders what would have happened if man had had only two stumps in place of his hands, or if everyone had been as the giant of Gath (II Sam. 21:20), who had six fingers and six toes.

Let us pause for just a moment and consider what is known as numerology,[3] e.g., certain superstitions that people have had for particular numbers being either lucky, or sacred. For example, 7 may be my lucky number; 13 may be unlucky for you. Much confusion on this account occurs in Greek and in Hebrew, where the letters of the alphabet may signify numbers also, and hence a whole word may mean a large number. Would you like to know the secret number of your principal, Mrs. Kent? In Greek her name is $\kappa\epsilon\nu\tau$; her number is 20 (κ) plus 5 (ϵ) plus 50 (ν) plus 300 (τ), all of which adds up to 375. One of the reasons given for the superiority of the Greek Achilles over the Trojan Hector was that Achilles' number was 1,276, whereas Hector's was only 1,225. In the New Testament the Greek word for beast (Rev. 13:18) signifies also the number 666. People have tried to guess for whom it was meant! Was it intended for the Roman Nero, or for some Catholic pope, or for the Protestant Luther? Remember that when someone says "I have your number," he may not necessarily mean your telephone number.

[3] G. B. Halstead, *On the Foundation and Technic of Arithmetic* (Open Court, Chicago, 1912).
W. F. White, *A Scrap Book of Elementary Mathematics* (Open Court; La Salle, Illinois, 1942).

Numbers and forms related

The study of numbers themselves can be fascinating.[4] Let us look at some numbers that bear the imprint of their shapes. First of all, there are *triangular numbers* like 1, 3, 6, 10, etc.; they can be represented in the form of triangles like ∴ for 3. The numbers 1, 4, 9, and 16 are *square numbers;* they can be arranged in the form of squares like :: for 4. Note that each square number is the sum of two triangular numbers ($4=1+3$, $9=3+6$, $16=10+6$, etc.). Furthermore, if you add all the odd numbers successively, you get for the sum in each case a square number, i.e., $1+3=4$, $1+3+5=9$, $1+3+5+7=16$. Finally, we have *cube numbers* which can be arranged in the form of a cube, e.g., $1\times1\times1=1$, $2\times2\times2=8$, $3\times3\times3=27$, $4\times4\times4=64$, as . The relationship between numbers and forms has been a favorite pastime of many people through the ages.

Another interesting number is what is known as a *perfect number.* In this case the number must equal the sum of all its divisors. For example, 6 has the divisors 1, 2, and 3; it is also their sum. It was for this reason that St. Augustine claimed that there had to be six days for a perfect creation. Twenty-eight has the divisors 1, 2, 4, 7, and 14, which add up to 28. Only twelve such perfect numbers are known. It is believed that there are no others, but as yet there is no proof. Let us consider next what are called *friendly numbers,* each of which is the sum of the other's divisors. For example, 284 has the divisors 1, 2, 4, 71, and 142, which add up to 220; whereas 220 has the divisors 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110, which add up to 284. One hundred such friendly pairs are known, but there may be more. A medieval prince once chose as his bride a girl whose name and his own formed friendly numbers; the marriage is supposed to have been a happy one.

Perhaps we are more familiar with *prime numbers* (each indivisible except by unity and itself) such as 1, 3, 5, 7, 11, 13, etc.; you can obtain them by what is known as the sieve of Eratosthenes.[5] How many prime numbers are there altogether? The Greek Euclid[6] gave a proof (cf. Book IX) in which he showed that there are an infinite number. How many would one expect to find in any given sequence of numbers? We do not know. We do not know in any million numbers whether there may be many or none at all. An even more unusual occurrence is the existence of twin primes, i.e., primes that are close together like 3, 5; 5, 7; 11, 13; etc. Do you know that some rules[7] for divisibility are based upon properties of prime numbers? For example, if the sum of the digits of a number is equal to a multiple of 9; the number itself has to be divisible by 9. Take 18; by adding 1 and 8, we get 9, which is certainly divisible by 9, and accordingly 18 must be divisible by 9. The Christian *Goldbach Problem,* dating from the eighteenth century, has become famous. It concerns even numbers as the sum of two primes. For example, 6 equals 3 plus 3; 8 equals 3 plus 5, etc. Under what conditions do such numbers exist? We do not know, although some persons have verified the rule up to 10,000.

Are any of you familiar with *Pythagorean numbers* such as the triplet, 3, 4, and 5? Take the square of each one, 9, 16 and 25, respectively. If we add the first two squares, we get the third square, i.e., $9+16=25$. Suppose we were to take 3^3 plus 4^3; would their sum equal some integer cubed? No! A French mathematician by the name of Pierre Fermat remarked in the seventeenth century that the sum of two cubes would never equal another cube. Such a relationship, indeed, does not seem to hold for any power higher than a square.

[4] E. T. Bell, *The Magic of Numbers* (New York, N. Y.: McGraw-Hill, 1946). R. Courant and H. Robbins, *What is Mathematics?* (London: Oxford University, 1941). L. Hogben, *Mathematics for the Million* (3rd ed.; New York, N. Y.: W. W. Norton, 1951). E. Kasner and J. Newman, *Mathematics and the Imagination* (New York, N. Y.: Simon and Schuster, 1943). A. Dresden and P. Noordhoff (trans.), B. L. Van Der Waerden's *Science Awakening* (Groningen, 1954).

[5] Bell and others, *op. cit.*

[6] T. L. Heath, *Euclid's Elements in English* (Cambridge University, 1926).

[7] Halstead and others, *op. cit.*

In the margin of Fermat's discussion of this theorem, he noted that he had a proof, but he failed to give it. To this day no one has yet been able to prove or disprove what is now known as *Fermat's last theorem.* It is possible, however, to find three squares which add to make a fourth square, for example, $2\times2+3\times3+6\times6=7\times7$.

Famous large numbers

One other interesting question! What is the largest number[8] that you can think of? A famous answer was given in terms of chess, which had been invented by a grand vizier for King Shirham of India. The grateful king granted the grand vizier's request for as much wheat as could be obtained by putting one grain on the first square of a chess board, two on the second one, four on the third, etc. How much wheat would be required if all the squares of the chess board were filled in this manner? The entire world crop of wheat for 2,000 years! Another famous answer was given by the Greek Archimedes.[9] It has been customary to think of the grains of sand as uncountable. He considered the number of grains of sand required to be piled up to the sky (the celestial sphere). He had to give the number in terms of a new unit, namely, 100,000,000 which he called an octade (actually a myriad myriads); he estimated it at 1,000 myriad octade octade octade octade octade octade octades. Nowadays we are much more accustomed to large numbers; for example, the number of atoms in the universe is about $3\times10\times10\cdots10$ (74 tens). There are many unsolved problems about numbers. I hope you will have fun trying to solve them.[10] The best method is still heuristic,[11] from a Greek word meaning to find. Just take some numbers and play with them—add them, divide them, square them, etc. Did you discover anything? If so, try some other numbers as a check. If not, try something else.

I would like to conclude with a few remarks about the invention of new numbers.[12] For a long time the only numbers that people knew were *integers,* which means untouched, i.e., whole numbers, like 1, 2, 3, etc. Man has found that he can invent new numbers by asking whether or not what he might be saying for some particular numbers might not be true for *any* numbers. For example, we know that 3×2 equals some number (in this case 6). We now ask, "Is there *any* number which when multiplied by 3 equals 2?" The answer is "yes"! You know the number; it is not a whole number, but a different kind of number which we call a *fraction,* i.e., $\frac{2}{3}$. Such fractional numbers can be shown to have all the properties of ordinary numbers. To be sure, we do have to be careful in using them; we must not speak about two-thirds of a girl, or three-fourths of a teacher. Another strange type of number is a so-called negative number. Here we start with the fact that 2 plus 3 equals some number (in this case 5). Now is there *any* number which added to 3 equals 2? Well, suppose we take away 1 from 3; the required number is thus minus one, i.e., -1; we call it a *negative number.* Is this a false number? Well, not exactly! If you look at the zero on a thermometer, you can regard the measure of the mercury below zero as really a negative number. To be sure, we have to be careful also in using such negative numbers; we could hardly speak of the negative age of a person. Let us consider one more instance of new numbers, those that are sometimes called irrational numbers. We start with the fact that 2×2 equals some number (which is 4). Is there *any* number which when multiplied by itself is equal to 2? Euclid,[13] who probably had it from the Greek Pythagoras, gave a very simple proof (cf.

[8] Gamow, *op. cit.*
[9] T. L. Heath (ed.), *The Works of Archimedes with the Method of Archimedes* (New York: Dover).
[10] W. W. R. Ball, *Mathematical Recreations and Essays*, rev. H. S. M. Copeter (New York: Macmillan Co., 1939).
[11] G. Polya, *How to Solve It* (Princeton, N. J.: Princeton University, 1957).

[12] A. N. Whitehead, *An Introduction to Mathematics* (New York: Henry Holt, 1911).
[13] Heath, *op. cit.*

Book X) that such a number would not be a whole number, not even the ratio of two whole numbers (i.e., a fraction). Hence the Greeks said that it is not rational, that it must really be *irrational*. The Pythagoreans were so disturbed that they spoke of it as an unutterable number; it seemed to them to be formless, not representable as a part of any line. Finally, let us note what some people call imaginary numbers. In this case, can we imagine *any* number which multiplied by itself equals -2? If so, we will call that an *imaginary number*. Such numbers, indeed, have been invented; they have all the properties of ordinary numbers. They have been particularly useful in science.

As we look back in history we see that man has truly invented numbers. Certain rules were restricted in practice, but they became more general by the introduction of new ideas. This is the story of all numbers; it is the story of all mathematics. It is not only an interesting story, but it has been a great boon for man; moreover, it is a story without an ending. The mystery of numbers began in fingerprints; the fingerprints are our own.

Subtraction by complement-addition-complement

The author presents this method of subtraction with the possibility of its being adopted experimentally by a few daring teachers.

This method, hereinafter called the *CAC* method, entails only one new concept, namely that of complementation with respect to nines. Algebraically, this concept is defined as follows:

$$X' = (10^n - 1) - X$$

where n is the number of digits in X.

Example:

$$2367' = 9999 - 2367 = 7632.$$

Notice that the complement can proceed from the left-hand digit as well as from the right-hand digit.

Finally, one can proceed to take the difference $(M-S)$ by finding $(M'+S)'$.

Example:

$$20038 - 6857 = (20038' + 6857)' = (79961$$
$$+ 6857)' = 86818' = 13181.$$

—Romae J. Cormier
Mathematics Department
Northern Illinois University
DeKalb, Illinois

Arithmetic: arthritis or adventure?

LUCY NULTON, *University of Florida, Gainesville, Florida*

Miss Nulton is a member of the P. K. Yonge Laboratory School faculty at the University of Florida.

Little Girl shook her head and swallowed a sob. "I don't know," she whispered and choked despairingly. "I don't understand." Father sat at her left, pages and pages of figures and a stubby pencil spread on the table before them. His face was troubled and sensitive. At her right, mother stood over her menacingly waving a long brown pencil.

"Don't understand!" exclaimed mother. "Don't understand! Haven't I just told you there's nothing to worry about understanding! Just memorize the rule and do what it tells you. You've seen your father work several problems of the same kind, and I've shown you over and over how the rule works. That's all there is to it. Now, take this next one!

" 'Just memorize the rule,' that's what teacher always says."

Little Girl grasped her pencil in cold, clammy hand and wrote the first number. A dull pain turned like a corkscrew up the length of her arm and her fingers stiffened. She stared at the arithmetic book without seeing.

"What do I do next?" she said dully.

"Write the divisor," instructed her mother.

"What's the divisor? Is it this one or is it that figure?" Little Girl tried to move her pencil to write and found her arm stiff with fear. Thirty years later she was to wonder if the arthritis which knotted her finger joints in awkward pain stemmed from that source. It felt the same, precisely like the arithmetic agony of her childhood.

If she could just understand what it was all about. If she could just know why you do the different things you do *to get the answer* and why numbers are made that way. Little Girl was a "why-ing" child. She was like that. She always had been. She knew why wild violets and lady's-slippers could be found in the woodsiest leaf mold of the Linkenfelter wood. She knew why squirrels grabbed off the biggest hickory nuts and raced with flag-waving tails to the big hole in the old oak tree. She and father had talked about that as they took long tramps through the wood and across the fields. Father could explain why the seeds of the yonkapin lay in the dried mud of the empty lake bottom and how they could be drilled to make rich, round beads for a necklace. He could explain how the drill worked and what friction was. Father could explain everything. Almost everything. Everything but arithmetic. It must have a "why," too, but nobody could tell her.

"Two-hundred-eighty-four," commanded her mother, who was an ex-teacher and knew all the rules.

Little Girl's hand ached. Her face was pinched and her eyes dark blue with terror.

"Why?" the word escaped. "Why do you write that?"

"Because that's the way you do it. That's why!"

Terror and memorization

Some nights they got through all the problems. More often the scene concluded with a command rustling her off to bed at the stroke of nine. Little Girl begged hysterically to be allowed to stay up until they were all done, all these examples for homework. Then she could memorize the figures while she dressed and ate breakfast. Then, maybe, when teacher stood her up at the blackboard she could get the answers and say the right figures. If she could, the big boys who knew the answers wouldn't look at her scornfully from the back of the room and teacher wouldn't make fun of her and rake her over with sarcasm. If she could get them all, her knees wouldn't ache while she stood at the board and her hand wouldn't drop the precious chalk. If she could just remember how the problems looked and see in her mind which figures went where. Little Girl was blessed with a good memory and this was a system that developed verbatim memory and rule-repeating parrotism. She managed to get good grades and a terror-ridden promotion each year. But fear is stronger than memory and bewilderment confounds reason.

Little Girl was the "Everychild" of that era, the epitome of all the little girls and little boys of the period. For them it was a period of confusion, sans reason, sans explanation; an era of learning by rote and acting by rule. Arithmetic was swallowed without meaning and applied without hope by countless numbers of children who grew numb with fear when arithmetic was mentioned and practiced in despair a memorized rule forced upon them by edict, tolerated only by frantic counting on concealed fingers to arrive at a preordained answer. Their hair may be gray now, but they still walk up and down the aisles of the supermarket, these children, loading their baskets with ignored prices or counting surreptitiously on fingers hidden under a bunch of celery.

Few understand numbers

To be sure, there were always some children who worked it out, understanding the meanings, enjoying the manipulations of numbers. Some found the system of numbers and knew it to be good. Some children, gifted in mathematical concepts, derived the "whys," just as some children read despite the a-b, abs, and despite the rote-reading which went stumbling, turn by turn, down the rows of seats. The readers figured out reading and read for fun, curiosity, and escape—escape from arithmetic. Just so, those gifted in numbers figured it out; but they were few.

So Little Girl grew up afraid, afraid of arithmetic and all its connotations. She was the child of parents who were victims of this old mode of teaching arithmetic. Teachers, too, feared arithmetic and wondered why these stupid children could not work it, but when they were confronted with a brazen child's "why?" they, themselves, could not answer anything more intelligible than, "Because that's the rule." For they also had been taught in the same way—rule by rule, memorization upon memorization, without meaning or understanding for generations.

Then, at last, the children's "whys" became more battering and more admissible. Those who understood arithmetic and saw it as challenging and as a system of thinking began to ask, "Well, *why* not? Why can't we teach arithmetic through meanings rather than by memorization?"

Now at last the scene is changing. For some years people have been studying a different kind of arithmetic teaching. Let us walk down the hall of one of these schools where teachers are attempting to teach arithmetic through meanings, and eavesdrop as we go.

"What is four?" asks a teacher of a group of happy seven-year-olds seated on the floor around her.

"It is one more than three."

"And one less than five."

"It is two and two."

"Four is three and one more."

"Or one and three more."

Answers come thick and fast with enthusiasm:

"Four is zero and four."

"Or four and zero. Zero is nothing."

"You can add zero and it's still four or take away zero and it's still four."

"You can divide four into two twos because two twos are four."

"If you had four and took two away you'd have two left."

"And if you took three away from four you'd have one left."

"If you didn't take away any, you'd still have four."

"If you had ten and took six away, there'd be four."

"And four is *always* four whether it is four ones or four tens or four oranges! It is four."

Thorough knowledge of "four"

These second graders surely have explored 4 with blocks, buttons, acorns, flannel boards, and "whys." They know it as a value in the number system and they use it in all arithmetical processes. Later in the day we overhear them making up problems about 4 in *their* lives, not about carpeting and wallpapering.

On down the hall two children in a third grade have just finished drawing a number chart on the board. Several children have charts of their own, constructed by measuring and illustrated with colorful, mathematical designs.

"What do you see in these charts?" asks the teacher, remembering that when she first started teaching, these number charts were used only to teach counting by ones, writing numbers, and counting by tens.

Eagerly, sparklingly, come quick insights into how our number system is built.

"The chart is ten wide and ten high."

"You can see that all the numbers are made of numbers from zero through nine. There aren't any others."

"Every row across the chart is from zero through nine."

"Yes, and every up-and-down row is ten more; it is like counting by tens."

"You call the up-and-down rows *vertical* rows," quietly intercepted the teacher.

"Yes, vertical; and across is *horizontal.*" This sounds like new vocabulary being aired.

"You can see that the number in ten's place on a horizontal row is the same each time."

"The number in one's place in a vertical row is always the same."

"If you look at the five's column you see that you add another ten to five ones each time."

"Three's column is the same way! Only you add tens to three ones."

"So is the four column. They all add tens to whatever ones are in the column."

"Say! A certain figure is in one's place and in ten's place. See? Take two: two is in one's place—why, it is in one's place ten times!"

"And it's in ten's place ten times," yelled Jimmy, beating Joe to his conclusion.

"So every figure is in the chart twenty times; ten and ten are twenty," Mary assumes.

There is rapid checking by everybody to find if this is true.

When all are satisfied the teacher asks, "What is the largest number on the chart that has two places?" Then, "What is the smallest number that has two places?"

The children pursue this rapidly from ten through the nineties, when some child announces, "The smallest two-place number is ten and the largest two-place number is ninety-nine."

"You can learn to count by threes on this number chart," points out David and proceeds to count.

"I can count by fours and by sixes and by sevens," boasts Tommy, then lamely trails off his boast with, "if I can see this number chart. Well, anyway, it's all there."

"And it's faster to count by groups than just one at a time," muses Ann.

"Do you see anything else on the number chart?" the teacher asks.

"Hey! Look how it goes crossways; eleven, twenty-two, thirty-three. Say!"

"Yea, looky! It's eleven more every time."

"Why?" Bill wonders.

There follows speculation and figuring. Soon Alice sees it: "Why, it moves over one every time, and it moves down ten every time; and one plus ten is eleven."

"Righto, Alice."

"O.K.! O.K.!"

"Bet it goes the same on the other crossways," shouts Danny.

"Try it and see. The crossway is called *diagonal.* Try the diagonal from nine to ninety. How does it go?" asks the teacher.

"But it doesn't go the same," in an aggrieved tone from several members of the group.

"Oh, wait! It does. No, wait! It goes by nines. Look! Nine more are twenty-seven."

"It does! It does!"

"It goes by nines."

"Whew! It's a system. The whole number chart's a system!"

"Yes," answers the teacher. "The number chart is a picture of our number system. Can you tell how our whole number system goes, as you say? Can you tell what it is all based on?"

"If you know your tens you can figure it out," volunteers Teddy.

"Yes, but you have ones till you get to ten. Oh, I see, it's from one to ten by ones and then you start all over with the tens and they go from one to ten, too." Philip was standing tall with the excitement of discovery.

"We've said that before," muttered John, "that counting goes by tens. Our number system is based on tens."

"Then can you tell why one of these diagonals moves by elevens and the other by nines?" the teacher asked.

This held them all for a few seconds. Then Kathy said tentatively, "It goes down ten every time. But it is going backward on the ones. So, if you go backward you take off one. Oh! One from ten is nine. That's why! That's why! It's back one and down ten, so it's nine every time."

Reasoning an adventure

Here was a group of children who, free to ask questions and explore for meanings, triumphantly reasoned out sixteen generalizations about our number system and did it in a spirit of adventure and discovery. The teaching consisted of guiding their thought processes, giving them terms, asking stimulating questions, and encouraging the hesitant. It also meant knowing the possibilities for learning which were embodied in the number chart and being unafraid to explore with the children. It meant one other crucial skill: knowing to which child to direct each question; which child was mature enough in number understandings to find that question challenging rather than fearsome. Not all children have the same abilities.

One more quality of teaching arithmetic through meanings is evident here. The children had experienced many of these generalizations before. They had experienced them concretely and repeatedly, year after year. It was upon this strength they could reason and compute.

Down the hall another group is making place-value pockets, measuring points to be folded, lettering names of places, cutting narrow strips of cardboard to be slipped into pockets. Here there is no dropping of pencil from numb, fear-stiffened fingers. A jolly laugh of discovery rings out from one who finished awhile ago. "Look! I thought it would be that way! If you put in ten tens you have a hundred, then you would have to put one in hundred's place to show those ten tens."

"Do you have to move it to hundred's place?" quietly inquires the teacher.

"No. It would still be ten tens."

Homework can be family fun

As we cross the playground toward our waiting car we pass a mother greeting a teacher.

"I've never seen so much adding in one brief evening as there was at our house," comments Susie's mother as she thrusts a potted plant into the teacher's hands. "The whole family had a great time with that puzzle. It even sharpened up my addition, playing with it."

Passing one more room we eavesdrop once more. "Wait a minute, John A. Give me time to figure that out and see if you're right," gasps the teacher who is a victim of the old way and must work her way through to a real understanding of numbers.

She is met with a merry laugh and excited approval from the children. They know she is not afraid to say in arithmetic as well as in science or any other subject, "I don't know, but I will find out." She meets the challenge without using the subterfuge: "What is the rule? Look it up in the text."

Yes, teachers are having to re-learn an arithmetic fraught with meanings about which one can dare to ask, "Why?" Fraught with confusions, too, for Little Girl, now a teacher, herself, must unlearn her fears, bewilderments, and dreads.

Teachers who are learning to teach children arithmetic through meanings feel that this is adventure, not arthritis, and that re-learning is well worth it when they hear a group of children exclaim with lilting voices, "This is fun" . . . "Arithmetic is fun" . . . "Arithmetic is like exploring. Let's do some more" . . . "Aw, not tomorrow! Now!"

The same name

The statement that "Mathematics is the art of giving the same name to different things" may appear to be entirely contrary to fact, but from a certain standpoint this statement conveys a very fundamental truth. It should be borne in mind that these different things must have in common the property to which this common name refers, and that it is the duty of the mathematician to discover and exhibit this common property.

—From "The Future of Mathematics" by G. A. Miller, *The Popular Science Monthly*, August, 1909.

Other number systems— aids to understanding mathematics

MARGARET F. WILLERDING
San Diego State College, San Diego, California

In addition to her responsibilities as associate professor of mathematics at San Diego State College, Dr. Willerding also serves as mathematics consultant for an NDEA project for junior high school teachers, both in San Diego County and San Diego city schools.

What is to be done with the student who reaches high school and is still unable to handle whole numbers and fractions satisfactorily?

Purpose and method of introducing other number systems

The needs of this student are easily overlooked since "the major part of the study of the skills and processes of arithmetic is completed in the seventh grade."[1] But Bryan also points out that

> The American school was designed to teach all the children of all the people, regardless of their physical, social, financial or mental endowments.[2]

Obviously, then, if we are to agree with this ideal, we are also forced to agree that a course for the mathematically retarded is just as appropriate as a course for the average or advanced mathematics student. As Robert Hogenmiller sums it up,

> There is need in the high school for a basic mathematics course for those students who have not mastered the fundamental operations with whole numbers, fractions and decimals.[3]

Hogenmiller goes one step further in his recommendation that such a course should be required of all students with such a deficiency. And if a student takes the course once without proper success, he should be required to take it again from another instructor.[4]

This second notion is apparently based on the idea that there is a certain minimum amount of arithmetic that everyone needs in order to live a constructive life, and that the school has an obligation to do whatever possible to teach this minimum level of understanding to all students. If we agree that this is a proper endeavor for a secondary school, we are then faced with the very difficult problem of how to go about it.

For several years these students have been exposed to all the standard teaching procedures in mathematics classes; and they have apparently gained very little. If they are required to sit through the same routine again, does it seem likely that they will have success next time? Is there, perhaps, some definite reason why all the effort and time spent so far on these students has been ineffectual? Perhaps a clue to the answers to these questions comes in a statement by Brueckner:

> Modern instructional procedures emphasize the importance of making arithmetic operations mathematically meaningful to children. Meanings are learned through a wide variety of planned experiences designed to help children to understand the number system and the ways in which it operates in the making of computations.[5]

Maybe the one absolute essential to the study of arithmetic is an understanding of how our number system works. Perhaps no amount of drill, or application to problems, regardless of how skillfully taught or motivated, can ever be effective with these students until they achieve an understanding of how our number system oper-

ates. It is on this premise that the following suggestions are being made.

The proposal is to begin a high school remedial mathematics class with a rather thorough study of our number system, using material that is new and strange to the students and for which they have no established bias. The essentials of the following systems would be covered over a period of several weeks:

Egyptian
Roman
Mayan
Hindo-Gabic
Binary

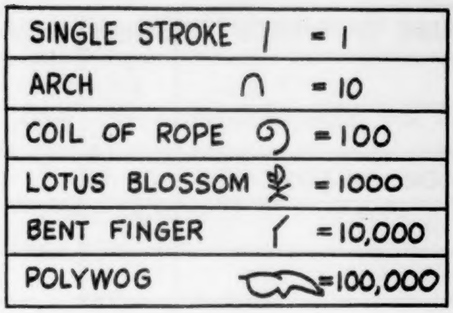
An attempt would be made to have the students generalize, through a series of student-discovery lesson sheets, that *any* number system consists of two essential factors: (1) symbols, and (2) rules for using the symbols. Then an attempt would be made to have the students analyze our Hindu-Arabic system and clearly decide what symbols and rules we use. That this might be a fruitful approach is indicated by Jones:

> Charts showing the same numbers written in different systems and others showing their differences, advantages, and disadvantages in writing and adding numbers are interesting, fun, and help show the essential characteristics of our own number system.[6]

Furthermore, it seems unnecessary to adhere to Jones's restriction of merely writing and adding. The student could, quite likely, discover many features of these other systems by attempting problems in the other fundamental operations and by attempting to designate fractional parts.

Teaching materials

While no attempt is made to build the complete unit, the following material is presented as an aid to the teacher. It should be complete enough to allow the teacher to expand the worksheet material to a usable length without going to additional sources.



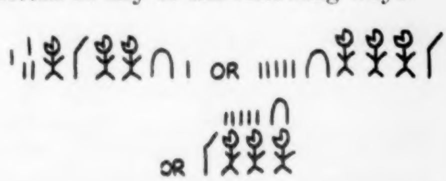
Egyptian Number System

Background information. The Egyptian hieroglyphic number system is based on the scale of 10. It is one of the simplest number systems, since it has no place-value concept associated with it; and its only rule is to add all the symbols in whatever order they occur. A very interesting fact about them is pointed out by Jones:

> The earliest written numbers [Egyptian hieroglyphics] which have survived until today are found on a mace which supposedly belonged to Egyptian king Nar-Mer who lived around 3400 B.C.[7]

The key to the numbers found on the mace, as well as to other Egyptian writings, is as follows:[8]

As an example,  represents our number 13,015. Although this number is written with the largest symbols to the left, it was more customary for the Egyptians to write in descending order from right to left.[9] Symbols were sometimes grouped without regard to order, even vertically, perhaps for the sake of artistic appearance.[10] For example, the above number could just as well be written in any of the following ways:



Student activities. 1. Complete the table on the next page.

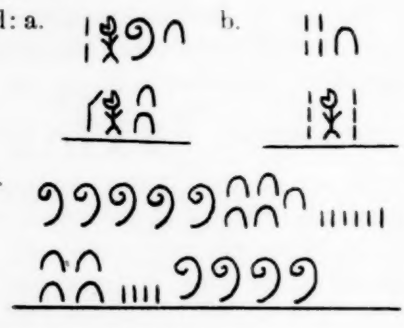
2. Write the Egyptian numbers from 1 to 20.

NAME OF SYMBOL	SYMBOL	VALUE
		1000
COIL OF ROPE		
		1

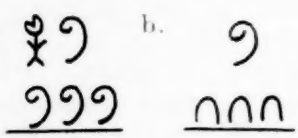
3. Write the equivalent of each of these Egyptian numbers:

a. b. c.

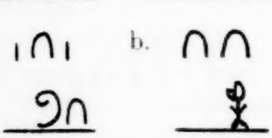
4. Write these numbers using Egyptian symbols:
   a. 90 b. 18 c. 243 d. 9009
5. Write each of these numbers in two ways, using Egyptian symbols:
   a. 6 b. 12 c. 33 d. 105
6. How would you write zero in Egyptian?
7. Add: a. b. c.



8. Subtract: a. b.



9. Multiply: a. b.



10. Which operations are easy and which are difficult, using Egyptian numbers?

Roman Numeral System

Background information. This ancient number system, which is still much used today, is more similar to ours than the Egyptian system, in that the Romans made some use of place-value. It is less similar to ours in that the base-10 idea is modified by a partial use of base 5.[11] This is made more clear if we arrange the Roman symbols as follows:

I = 1	V = 5
X = 10	L = 50
C = 100	D = 500
M = 1000	

In the designation of large numbers, a horizontal bar placed over a letter increased its value by a factor of 1000;[12] e.g., $\overline{\text{XI}}$ represents 11,000. Even so, the Roman system was not always very concise; e.g., 9,989 would have been written MMMMMMMMMDCCCCLXXXVIIII.[13]

The subtractive principle is the feature of the Roman system that is slightly analogous to our place-value. When a symbol for a smaller unit was placed before a larger unit it meant the difference of the two units. However, the subtractive principle was only sparingly used in ancient and medieval times. The fuller use of this principle was not introduced until modern times.[14] Using this idea, the example above is considerably shortened; i.e., 9,989 can be written M$\overline{\text{X}}$CMLXXXIX. In this connection Friend relates:

> We retain today the subtractive principle when we state the time of day as so many minutes to (make up) the hour. Thus we usually say, for example, it is five to ten instead of 55 minutes past nine. . . . Similarly we seldom say three-quarters past two; we prefer to say a quarter to three.[15]

Regarding other operations, "ancient multiplication was a matter of repeated addition. Division, even in early times, probably was done by means of repeated subtractions."[16] Or as Friend says, "For mere enumeration the Roman system was both simple and clear, but for the more serious operations of multiplication and division it was entirely unsuited."[17] Thus a Roman schoolboy could not use the method of multiplication that we use. A

very simple problem, such as 3 times 366, would have required something like the following:

C C C L X V I
C C C L X V I
C C C L X V I
M X C V I I I

Pedagogically, it seems advisable first to present the Roman numerals as the ancients used them—with the subtractive idea minimized. Then, later, we use the subtractive idea in a maximum way to shorten numbers as much as possible. This should emphasize for students the necessity of having clear rules in a number system, as well as convenient symbols.

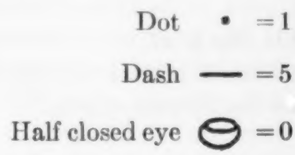
Student activities. 1. Write the symbols of the Roman system and the value of each.

2. Give the Roman numerals from 1 to 25.
3. Write these numbers using Roman symbols: a. 47 b. 109 c. 5240
4. Write these Roman numerals in our symbols: XLI, LXIX, MCDIV.
5. Without using the subtractive principle write these numbers in Roman symbols: a. 949 b. 717 c. 444
6. Write these numbers in Roman numerals, using the fewest symbols possible: a. 1999 b. 9429 c. 37562
7. Multiply 4 times 77 the way a Roman student would have done it.
8. Divide 88 by 22 the way a Roman student would have done it.
9. Give an important historical event that happened in the year MDCCLXXVI.

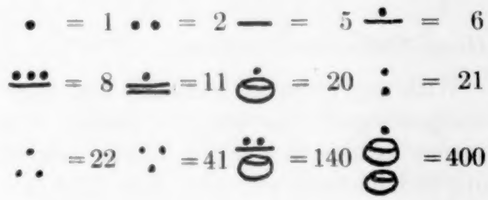
Mayan number system

Background information. An early Indian tribe of Central America, the Mayas, developed a very adequate number system quite similar to ours. Because of the ease with which computations could be made in this number system, Mayan astronomers were able to compute the length of a year and predict the motion of comets almost as accurately as we can today. In fact, this system might very well have been in great use today had not Mayan culture been destroyed during the conquest by the Spaniards.[18]

The Mayan number system was very similar to ours, and the place-value idea completely developed, despite the fact that only three symbols were used. They were:

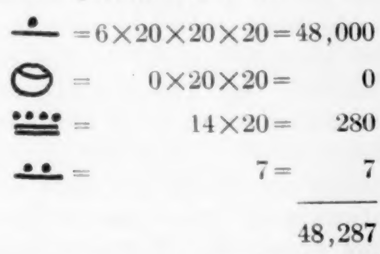


Surprisingly, considering the choice of symbols, the principal base of the system was 20, with the variation in the 2nd place of 18 to give a closer approximation of the length of a year for calendar purposes.[19] However, for pedagogical reasons with slow learners, it seems advisable to assume a strict base-20 notation. Then:



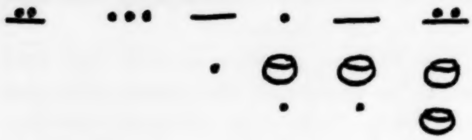
Any number, large or small, can be compactly written. For example, 48,287 is represented:

This is interpreted as follows:



Student activities. 1. Write the Mayan numbers from 1 to 25.

2. Write these numbers in our symbols:



3. Write these numbers in Mayan symbols:
   a. 148 b. 962 c. 47
4. Do you think printers would like it if we used the Mayan system?
   Why?
5. Add: 
6. Subtract: 

7. What feature of the Mayan system permits such compact enumeration with only three symbols?

Hindo-Gabic number system

With any group of students, and especially so with slow learners, humor is a potent weapon in creating an atmosphere in which learning will take place. This system is introduced because it offers an opportunity to combine humor with sound principles.

Background information. The Hindo-Gabic[20] system is the creation of Dr. J. E. Eagle. It is identical with our Hindu-Arabic system except that it has a base of 4 rather than a base of 10. The names of the symbols give the humor to the situation and require a dramatic introduction for full effect. As the story goes, this is the number system of a primitive Australian tribe who liked dogs; and their number system developed around the fact that a dog has four legs. The symbols, values, and names are shown in the next column.

Student activities. With this system all operations are almost the same as the operations in our system, but the students will not expect this. Therefore, it is an ideal system to use to redevelop the ideas of carrying and borrowing in fundamental operations. Pedagogically, this seems like an ideal place to use the student-discovery technique. Without mentioning the close parallel to our system, give the students many exercises that require carrying and borrowing. Then let them work at them until they discover for themselves the parallel to our system. Because of the strangeness of the system and symbols, the students will be required to think very carefully about the meanings of "carry," "borrow," and "place-value." And this is precisely the desired outcome.

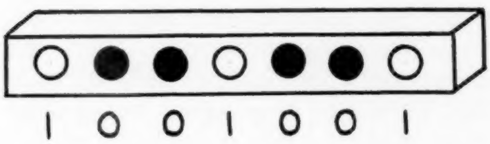
Hindu-Arabic Number	Hindo-Gabic Number	Hindo-Gabic Number Name
0	0	Zero
1	1	One
2	2	Two
3	3	Three
4	10	Doggy
5	11	Doggy one
6	12	Doggy two
7	13	Doggy three
8	20	Twoggy
9	21	Twoggy one
10	22	Twoggy two
11	23	Twoggy three
12	30	Throggy
13	31	Throggy one
14	32	Throggy two
15	33	Throggy three
16	100	One *houndred*
17	101	One houndred one
. .		
20	110	One houndred doggy
. .		
24	120	One houndred twoggy
. .		
64	1000	Kennel

Binary number system

The Binary system is another self-motivating and interest-creating system and can be conveniently used to redevelop the meaning of fractions.

Background information. The binary system uses only two symbols, 0 and 1. It is identical with our system and with the

Hindo-Gabic, except that it has a base of two, as indicated by its name. It has widespread application in the field of modern research, since digital computers,[21] such as Univac,[22] use binary digits. Although they could be built, "it is easier to design high speed digital computers to perform binary arithmetic than decimal arithmetic."[23] An oversimplified explanation of the operation of these computers is that they are electronic machines and that "0" represents no contact or no current flow and "1" represents a contact or a flow of current. As a demonstration of this point, Willerding[24] suggests a very simple device: a box, such as a shoe box, with holes in the bottom and with Christmas tree lights (parallel-type) fitted into the holes. Turning different combinations of the bulbs on and off gives any desired binary number. For example, the number seventy-three would be represented as:



A simple explanation of how fractions are written is:

The binary point in a binary number is the point which marks the place between positive powers of two and negative powers of two, and is of course analogous to the decimal point in a decimal number. Thus 100.101 means four, one half, and one eighth or four and three eighths.[25]

Student activities. As a novel introduction to binary arithmetic it has been suggested that students be asked to interpret such statements as 11+10=101, or the story about the eccentric mathematician.[26] When he died, his friends found the following note in his handwriting,

I graduated from college when I was 10,111 years old. A year later, I, a 11,000-year-old young man, married a 10,011-year-old young girl. Since the difference in our ages was only 101 years we had many common interests and hopes. A few years later we had a family of 100 children. I had a teaching job, and my salary was 11,000,000 dollars a month.

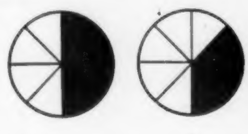
And, as with Hindo-Gabic numbers, it would seem beneficial to have students work problems involving all the fundamental operations, especially ones involving carrying and borrowing. To make the work with fractions as concrete as possible, such problems as the following are suggested.

1. Consider the numbers given to be binary numbers and shade the indicated portion of each group of circles:

1.01

2.101

2. Using binary numbers, tell what part of each circle is shaded.



a. b.

Summary activities. From this unit it is hoped that the student will gain greater

Ours	*Egyptian*	*Roman*	*Hindo-Gabic*	*Binary*
			314	
		V		
				1011

understanding of how our own system works. The nature of the summary activities should cause the students to compare our system with the others. For example:

1. Complete the table on page 355.
2. Assuming you are talking to someone who has never used any number system except ours, give the symbols and rules required for working with each of the systems we have studied.

Notes

1. John C. Bryan, "Mathematics In General Education," *School Science and Mathematics,* LVIII (April, 1958), 251.
2. *Ibid.*, p. 250.
3. Robert E. Hogenmiller, "A Science and Mathematics Curriculum for Terminal Students in High School," *The Bulletin of the National Association of Secondary School Principals,* XLII, 240 (October, 1958), 109–15.
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Math Pen Pals—

Girls and boys in the mathematics classes at Ridgewood Junior High School would like to become Math Pen Pals with other junior high students. Introduce yourself, describe your home, tell about your interests, and then tell about work being done in your mathematics classes.

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222 Ridge Road East
Rochester 22, New York

Some basic geometric ideas for the elementary teacher

LEON RUTLAND, *University of Colorado, Boulder, Colorado*

MAX HOSIER, *State College of Iowa, Cedar Falls, Iowa*

Dr. Rutland is a member of the Department of Mathematics at the University of Colorado, and Dr. Hosier is a member of the Department of Teaching at State College of Iowa. Both have been members of the 1960 and 1961 Writing Teams for the School Mathematics Study Group Project on Elementary School Mathematics.

More and more we speak of the *mathematics program* instead of the arithmetic program of the elementary school. We do this because the content of the program places emphases not only on arithmetic but also on geometric understandings and skills. Thus, just as teachers need to be more familiar with the present-day approaches to arithmetic content, so it is that they need to be familiar with present-day approaches to geometric content. Yet, not only the approach to arithmetic but also the approach to geometry is different today than it was some years ago. More specifically, today we believe it helpful to think of basic geometric ideas in terms of sets of points. What does this mean? It is the intent of this article to show how basic geometric ideas are interpreted from this point of view.

Points and curves

If we look at the definition of a word in the dictionary, we will find this word described in terms of other words. If we do not know the meanings of the words used in the definition, we look up the meaning of these words. Continuing this process, we are eventually led back to the word with which we have started.

To avoid this "circle of definitions" in mathematics, some terms are not defined but are described by properties. Two of the undefined terms are "point" and "line," and one of their properties is that given two points, there is only one line which contains both points.

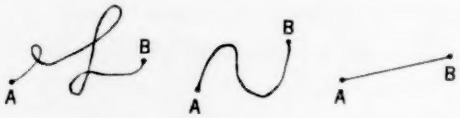
We think of a point as being so small that it has no size at all. No pin has a sharp enough point to "pinpoint" the exact location of a point. We have never seen a point. Yet, we would like to draw pictures of certain collections of points. We do this by drawing dots on a sheet of paper. Each dot actually covers more points than we could ever count, but the picture helps us to talk about the point. The dots we make represent or "stand for" points.

The child in the elementary school may think of a point as an exact location. Since a point is a fixed location, it does not move. If a dot is made on a sheet of paper to mark a certain point in space and the sheet of paper is then moved, the dot marks a different point or location. If we think of every location on the earth, in the earth, in the sky, in the universe, in every object on the earth, and so on, as occupying points in space, then we have some idea of points as locations.

Space can now be described as the set of all possible locations, that is, space is the set of all points. A collection of points is called a set of points.

Before trying to describe a line, let us describe a curve between two points

(locations) in space. A *curve* is the set of points passed through in going from one of the points to the other. It is, therefore, a particular set of points in space. Here are some pictures of curves from the point A to the point B:



(a) (b) (c)

Figure 1

Figure 1(c), drawn with the aid of a ruler, is the most direct path and is called a *line segment*. Thus, in mathematics, a curve does not have to be "curved" as we commonly think of it, because a curve is any set of points passed through in going from one point to another. We use the symbol $\overline{AB}$ to indicate the line segment joining point A and point B. The mark on the paper is just a representation or model of the line segment. The line segment is the set of points.

If we think of a line segment as being extended without end in each direction, we have the concept of a *line*. Its representation is shown in Figure 2. The arrows show that it does not end.

Figure 2

In a similar manner, if we think of a table top or any flat surface being enlarged without end, we have the concept of a representation of a set of points called a *plane*. If we take any two points of the plane and draw a line through them, every point of the line will also be a point of the plane. Thus, a plane contains more lines than can ever be counted (an infinite number).

Many other relations concerning points, lines, and planes can be observed. Following are some examples.

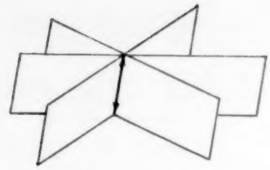


Figure 3

Through a line in space there are an infinite number of planes. See Figure 3.

Through three points not of one line there is only one plane. See Figure 4.

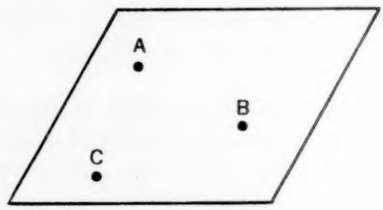


Figure 4

If two different lines in space intersect, their intersection is one point. See Figure 5.

If a line and a plane intersect, either the intersection is just one point (Fig. 6) or else every point of the line is a point of the plane (Fig. 7).

If two planes intersect, the intersection is a line (Fig. 8).

Figure 5

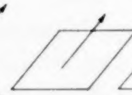


Figure 6

Figure 7

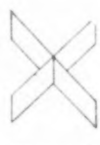
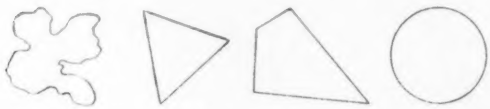


Figure 8

In the preceding paragraph, the word "intersect," as applied to the sets of points, means just what the common usage of the word would imply. The intersection is the set of points common to the sets.

When with a pencil we trace a curve on a sheet of paper, we are representing a curve which lies in a plane. If the curve begins and ends at the same point, but otherwise does not intersect itself, we obtain a *simple closed curve*. In this article

we are discussing only simple closed curves in a plane. Some pictures of simple closed curves are shown in Figure 9.



(a) (b) (c) (d)

Figure 9

A *polygon* is a special type of simple closed curve which is composed of line segments. Drawings (b) and (c) of Figure 9 illustrate polygons. A *triangle* is a polygon composed of three line segments and a *quadrilateral* is a polygon composed of four line segments.

When a set is composed of two or more subsets, we say it is the *union* of the subsets. Thus, a triangle is a simple closed curve which is the union of three line segments.

A simple closed curve consists only of the points of the curve. The points in the interior of a simple closed curve are not points of the curve. We speak of the points of the simple closed curve and the points in the interior of the curve taken together (that is, the union of the two sets) as a *plane region*. Thus the set of points of a triangle together with the set of points in the interior of the triangle form a triangular region. When we find the "area of a triangle," we are actually finding the area of the triangular region, since the triangle itself has no area.

In order to describe angles, we need the idea of a ray. A *ray* is the part of a line consisting of a point on the line called the *endpoint* of the ray and all points on the line in one direction from the endpoint. In drawing a picture of a ray, we will indicate that a ray extends indefinitely in only one direction by means of an arrowhead as is shown in Figure 10.

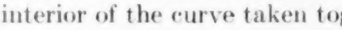


Figure 10

(a) (b) (c)

Figure 11

An angle is a set of points consisting of two rays not on the same line and which have the same endpoint. Angles might be depicted as shown in Figure 11.

Surfaces

In Figure 12, let point D be a point not in the plane of triangle ABC. Line segments $\overline{BD}$, $\overline{AD}$, and $\overline{DC}$ form, with the original triangle, three new triangles. The union of triangle ABD and its interior is a triangular region (shaded in Figure 12).

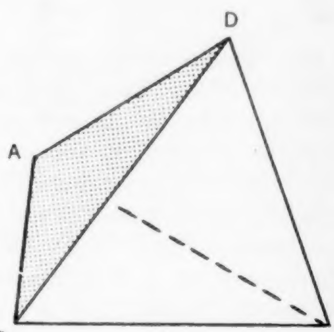
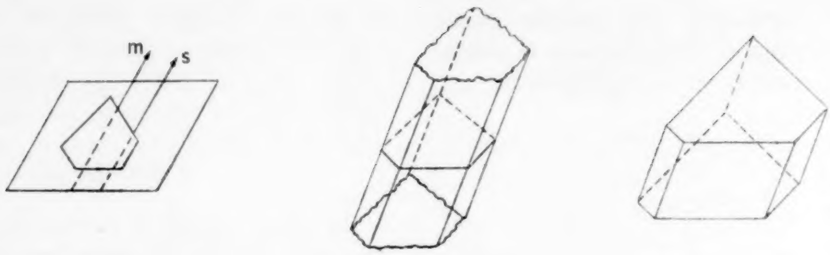


Figure 12

Triangular regions are also formed by triangle BDC and its interior, triangle ADC and its interior, and triangle ABC and its interior. The union of these four triangular regions is an example of a simple closed surface. This particular simple closed surface is called a pyramid. The pyramid is composed just of the plane regions—that is, it is "hollow."

A simple closed surface divides space, other than the set of points on its surface, into two sets of points, the set of points interior to the simple closed surface and the set of points exterior to the simple closed surface. One must pass through the simple closed surface to get from an interior point to an exterior point.



(a) (b) (c)

Figure 13

In a plane, we called the union of a simple closed curve and points in its interior a plane region. In a similar manner, we will call the union of a simple closed surface and points in its interior a *solid region.* A plane region has area, and a solid region has volume.

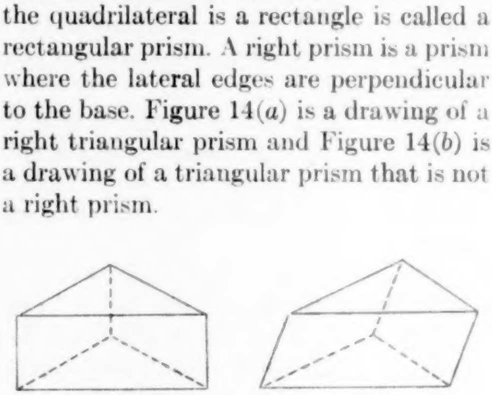
Prism

The first simple closed surface to be discussed is a *prism.* Think of a polygon and a line *m* not in the plane of the polygon. Suppose the line *m* intersects the polygon. Think of another line *s* that also intersects the polygon and is parallel to line *m* as shown in Figure 13(a). The union of all lines, such as *s,* that intersect the polygon and are parallel to *m,* is a surface such as is shown in Figure 13(b). The surface extends infinitely up and down.

Think of the surface being cut by two parallel planes. The resulting simple closed surface, as shown in Figure 13(c), is called a *prism.* Each of the plane regions is called a *face* of the prism. The two faces formed by the parallel planes are called the *bases.* The faces other than the bases are called the lateral faces. The intersection of two faces is a line segment called an *edge.* The intersection of two lateral faces is a lateral edge.

The prism is the union of the faces. A prism is a surface. It is "hollow." Points in the interior of the prism are not points of the prism.

If the polygon outlining the base is a triangle, the prism is called a triangular prism. A prism is quadrangular if the polygon is a quadrilateral, and it is pentagonal if the polygon is a pentagon. The special quadrangular prism in which the quadrilateral is a rectangle is called a rectangular prism. A right prism is a prism where the lateral edges are perpendicular to the base. Figure 14(*a*) is a drawing of a right triangular prism and Figure 14(*b*) is a drawing of a triangular prism that is not a right prism.



(a) (b)

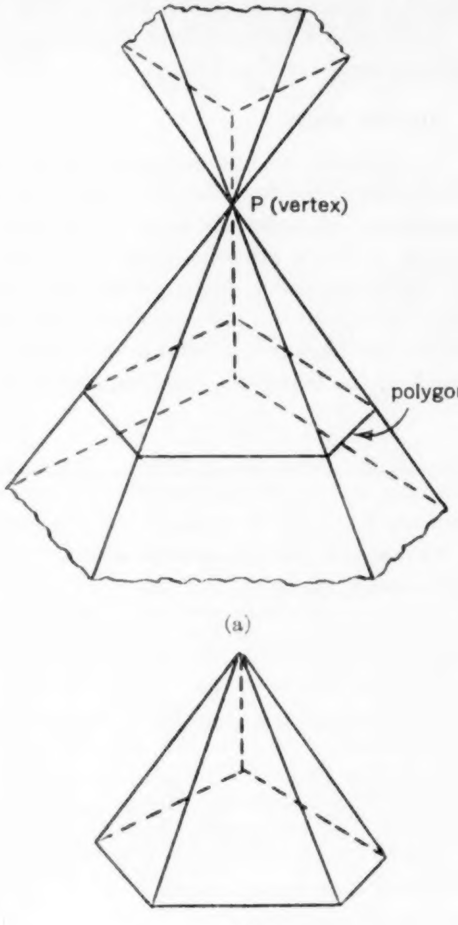
Figure 14

Pyramid

Consider a polygon and think of all lines that intersect the polygon and also pass through a point *P* (called the vertex) not in the plane of the polygon. The union of all these lines is a surface such as is shown in Figure 15(a). The surface consists of two parts separated by the vertex. Each of these parts is called a *nappe.* A *pyramid* is a simple closed surface formed by part of one of the nappes and a plane that cuts it. See Figure 15(b).

Using the same classification we used for the prisms, a pyramid is called triangular, quadrangular, pentagonal, etc., according to whether the polygon outlining the base is a triangle, quadrilateral, pentagon, etc. See Figure 16.

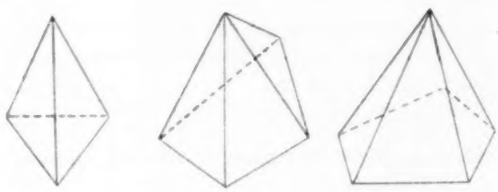
A simple closed surface consisting en-



(a)

(b)

Figure 15



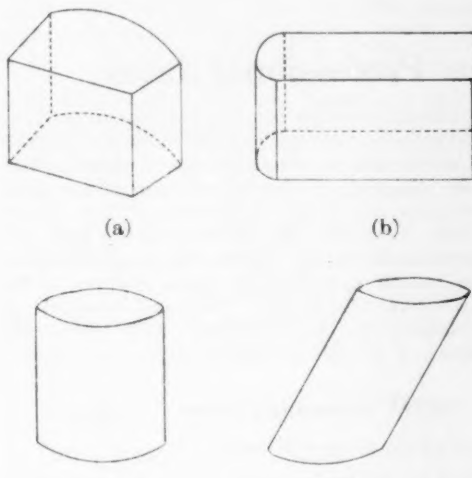
(a) (b) (c)

Figure 16

tirely of polygonal regions is called a *polyhedron*. The prism and the pyramid are examples of polyhedrons. The triangular pyramid, for example, is also called a *tetrahedron*. The remaining simple closed surfaces to be discussed are not polyhedrons.

Cylinder

A cylinder may be defined in a manner similar to the way we defined a prism. In fact, a prism may be thought of as a special case of a cylinder. Start with any simple closed curve (it could, of course, be a polygon). Construct a simple closed surface exactly as was done for the prism. The surface is called a *cylinder*. Some examples of cylinders are shown in Figure 17. Figure 17(*c*) is a drawing of a right circular cylinder. Figure 17(*d*) is a drawing of a circular cylinder that is not a right cylinder.



(a) (b)

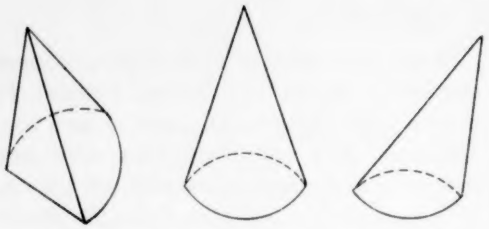
(c) (d)

Figure 17

Cone

What do you think the simple closed surface would be if we replaced "polygon" by "simple closed curve" in the definition of a pyramid? Here, however, we will place the restriction that the simple closed curve be convex. A simple closed curve is convex if a line segment joining any two interior points lies entirely within the curve. This surface is a *cone*. A pyramid, then, is just a special type of cone. Some examples of cones are shown in Figure 18.

A *circular cone* is obtained when the simple closed curve is a circle. A right cone



(a) (b) (c)

Figure 18

is obtained when the altitude is perpendicular to the base. Figure 18(*b*) is a drawing of a right circular cone. Figure 18(*c*) is a drawing of a circular cone that is not a right cone.

Further study

It certainly is not necessary that an elementary teacher be an expert geometrician in order to help elementary-school children effectively develop some of these geometric ideas intuitively. On the other hand, no one can deny that the better the background and understanding one has, the better his teaching should be.

Professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE ARITHMETIC TEACHER. Announcements for this column should be sent at least two months prior to the month in which the issue appears to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

Fortieth Annual Meeting

April 16–18, 1962
Jack Tar Hotel, San Francisco, California
Kenneth C. Skeen, 3355 Cowell Road, Concord, California

Joint Meeting with NEA

July 4, 1962
Denver, Colorado
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D.C.

Twenty-Second Summer Meeting

August 23–25, 1962
University of Wisconsin, Madison, Wisconsin
H. Van Engen, School of Education, University of Wisconsin, Madison 6, Wisconsin

Other professional dates

Texas Council of Teachers of Mathematics

November 2–4, 1961
University of Texas, Austin, Texas
Dr. Robert E. Greenwood, Mathematics Department, University of Texas, Austin, Texas

Northeastern Ohio Teachers Association

November 3, 1961
Cleveland Engineering and Scientific Center, 3100 Chester Avenue, Cleveland, Ohio
Walter F. Rosenthal, 481 Northfield Road, Bedford, Ohio

Virginia Education Association, Mathematics Section

November 3, 1961
Womens' Club Auditorium, Richmond, Virginia
Simeon P. Taylor, Yorktown High School, 5201 N. 28th Street, Arlington 7, Virginia

Georgia Mathematics Council

November 10–11, 1961
Rock Eagle State Park
Eatonton, Georgia
Martha Rogers, 2908 Macon Road, Columbus, Georgia

New York Society for the Experimental Study of Education; Section 10—Mathematics

December 1, 1961
Washington Irving High School, New York, New York
Mary G. Rule, 58 Spring Avenue, Bergenfield, New Jersey

Women's Mathematics Club of Chicago and Vicinity

December 2, 1961
Henrici's Restaurant, 71 W. Randolph, Chicago, Illinois
Dr. Ruth Ballard, University of Illinois, Navy Pier, Chicago 11, Illinois

The Mathematics Club of Greater Cincinnati

December 2, 1961
Roy D. Matthews, Board of Education, 608 E. McMillan Street, Cincinnati 6, Ohio

Classroom climate and the learning of mathematics

FRED GUGGENHEIM

Board of Education of the City of New York, New York City, New York

Dr. Guggenheim is research associate with the New York City Bureau of Curriculum Research. He also teaches an in-service course in curriculum research for teachers.

In recent years, particularly with the advent of educational television and the United States policy of emphasis on education, the evaluation of the teacher's role in the classroom has moved into the public spotlight.

This investigation of the effect of classroom climate on learning grew out of an attempt to help resolve the question of the teacher's effectiveness in teaching an "academic subject." The critics of contemporary education claim that today's schools emphasize "social" and "cultural" skills to the detriment of the basic "three R's."

Until the last two decades, there was little attempt made to study the interactive processes that take place in a classroom. The bulk of the research in teacher effectiveness was aimed at isolating single elements or characteristics which were hypothesized to be crucial in determining effective teaching. Comparisons of similar studies often led to contradictory results. Part of this conflict results from failure to use established psychological or sociological theory [3].* The same limitation found in an examination of theory related to teacher competence is found in the use of criteria. Either a lack of precise definition of terms and concepts or a reduction of complex theory to suit the design prevails. This multitude of ambiguity and confounding of results indicates a fresh and new approach is needed for the assessment of teacher competency [4, 7, 1].

To appraise the problem adequately, and within limits, it was necessary to select an academic area which would be least likely to be affected by the "social" and "cultural" influences of teachers. The area of mathematics seemed the most suitable because of the large amount of drill needed in the elementary school and the relatively nonverbal nature of the subject.

* Numbers in brackets refer to the References at the end of this article.

Purpose

The purpose of this study was to investigate the influence that dominative and integrative classroom climates have on the learning of mathematics by third-grade children. The Wrightstone Teacher-Pupil Rapport Scale [8] was utilized to evaluate classes for selection for the two groups.

Procedure

Because of practical limitations, it was necessary to use an indirect method of evaluating and selecting teachers for inclusion in the study. It was assumed, however, that there was a high correlation between teacher behavior and classroom climate.

Dominative climate was defined as one in which the majority of the behaviors and interactions within the classroom were

Table 1

Distribution of scores on the Wrightstone Scale according to social climate of the classrooms

	Dominative group												*Integrative group*											
	Teacher												*Teacher*											
Category	1	2	3	4	5	6	7	8	9	10	11	*Total*	1	2	3	4	5	6	7	8	9	10	11	*Total*
(3)* 1	2	2	2	2	2	2	2	2	2	2	2	22	3	3	3	3	3	3	3	3	3	3	3	33
(5) 2	2	1	2	2	1	2	2	1	1	2	1	17	4	3	4	3	3	4	3	4	4	3	3	38
(5) 3	2	2	2	3	3	3	2	2	2	3	1	25	5	3	4	4	3	4	3	4	4	4	4	42
(4) 4	2	3	3	2	3	3	3	2	2	3	3	29	5	4	5	4	4	3	4	5	4	4	4	46
(5) 5	1	1	2	2	2	2	1	1	1	2	1	16	5	3	4	4	3	4	3	5	4	4	3	42
(3) 6	2	3	3	2	2	2	2	1	2	2	2	23	3	3	3	3	3	2	3	3	3	3	3	32
(5) 7	1	1	2	2	2	2	2	1	2	2	1	18	5	4	4	5	4	3	3	4	4	4	4	47
(4) 8	1	1	1	1	2	1	1	1	1	1	1	12	3	3	2	3	3	3	3	3	3	3	3	31
(4) 9	1	1	2	2	2	2	2	1	1	2	1	17	4	3	3	4	3	3	3	4	3	3	3	36
(5) 10	1	1	2	2	2	2	2	1	2	2	1	18	4	4	3	4	3	3	3	4	3	3	3	37
Total	15	16	21	20	21	21	19	13	16	21	14		41	33	35	37	32	32	30	39	35	34	33	

* Numbers in parentheses indicate number of items per category.

teacher-centered. In the dominative climates, the teachers were usually aggressive or irritable in their behavior toward the pupils and generally got things done by coercion. The pupils in the dominative group were fearful, and the group was restless.

An *integrative climate* was described as having reciprocal pupil-teacher interaction. A maximum of warmth was present in the classroom, and the pupils enjoyed the work. In the integrative group, the teacher was warm and sympathetic.

The Wrightstone Scale consists of ten related categories or subtests. Each category can be rated numerically from one to three, four, or five, depending on the weight of the category in the total scale. The maximum score obtainable in the Wrightstone Scale is forty, indicating a highly integrative climate, and the lowest score obtainable is ten, indicating a highly dominative climate. The ten categories of the scale are: (1) Pupil-Teacher Interaction Pattern, (2) Degree of Social Interaction, (3) Quality of Social Interaction, (4) Interest, (5) Enjoyment, (6) Role Structure, (7) Emotion of Leader, (8) Teacher Orders or Suggests, (9) Physical Tension of Group, (10) Emotion of Pupil Group.

The distribution of classroom scores is presented in Table 1. The reliability of the scale and the raters was established by having two persons rate twenty-two classes independently. A rank difference coefficient of correlation of .83 was obtained, which was similar to the reliabilities reported by other investigators [8, 5].

The procedure used to investigate the problem was to determine the initial mathematics achievement level of the pupils at the beginning of the school year, match the groups in mathematics achievement, and retest the same groups after a year of instruction in the different climates. The criterion of climate effectiveness was the scores attained on retesting. The New York Inventory of Mathematical Concepts, Grade 2, was used in the initial testing and matching of groups. The Grade 3 version was used in the final evaluation.

The pupils were also matched by intelligence test scores, using the Otis Quick-Scoring Mental Ability Tests: New Edition, Alpha Short Form.

The selection of schools used in the study was made from a list of all public schools designated as average by the Elementary Division of the Board of Education of the City of New York. Eleven

Table 2
Characteristics of the pupils in the matched groups

Group	*Mental age*			*Chronological age*			*I.Q.*			*Mathematics achievement*		
	Mean	*S.D.*	*t*	*Mean*	*S.D.*	*t*	*Mean*	*S.D.*	*t*	*Mean*	*S.D.*	*t*
Integrative	115.50	10.75	.09	103.52	3.51	.24	110.90	10.50	.00	38.09	4.17	.00
Dominative	115.40	11.05		103.25	3.49		110.90	10.50		38.09	4.17	

schools were selected from which the twenty-two classes were drawn. Fifty third-grade classes in these eleven schools were evaluated with the Wrightstone Scale and ranked in order of rating. The highest eleven (Integrative) and lowest eleven (Dominative) classes were selected for the experimental and control groups. The significance of the difference between the two groups in classroom climate was established by the t test ($t=8.63$).

An evaluation of the distribution of pupils in the two groups was made to determine if the mean I.Q. and initial mathematics achievement scores of the pupil population were evenly distributed. It was found that there was a significant difference between the Integrative and Dominative groups in I.Q. and initial mathematics achievement. The groups were matched by pooling the classes within each group and adjusting their means and variabilities by dropping certain pupils so that the difference between them became nonsignificant. The total number of pupils dropped was 206, leaving 195 in each group. At the end of the school year, after the final adjustments in matching were made, there remained 156 pupils in each group. The characteristics of the pupils in the matched groups are presented in Table 2.

To achieve greater accuracy in the evaluation and to discover if there were any differential effects of climate on level of achievement, the pupils in each group were subdivided into three levels of initial mathematics achievement. The selection of level was made by dividing the total pupil population of this study into three groups and designating each of them as representing high, average, and low levels of mathematics achievement in relation to the distribution of the whole group.

The evaluation of the experiment utilized a treatment-by-levels design in which the tests were administered to two matched groups. The final mathematics achievement scores for the pupils were tabulated in a double-entry table of rows of levels, columns of climate, and cells to the subgroups within the three levels. An analysis of variance technique was used to evaluate the data. The total sum of squares was analyzed into four parts. The total sum of squares consisted of a sum of squares for climate, a sum of squares for levels of achievement, a sum of squares for climate by levels of achievement, and a sum of squares for within subgroups.

Results

The hypotheses of this research were tested with the F test, using an analysis of variance technique. A treatment-by-levels design was used to test the following hypotheses:

1 The social climate of a classroom does not affect the learning of mathematics by third-grade elementary-school pupils, and pupils in integrative climates will make no greater achievement in mathematics than do pupils in dominative classroom climates.

Table 3
Analysis of variance

Source of variation	*Degrees of freedom*	*Sum of squares*	*Mean square*	*F*
Treatments (Climate)	1	113.29	113.29	1.55
Levels (Achievement)	2	1,643.20	821.60	
Cells	(5)	(1,802.96)		
Interaction				
(Climate by achievement)	2	46.47	23.23	
Within subgroups	306	22,265.92	72.76	
Total	311	24,068.88		

2 The amount of achievement for pupils average in achievement in integrative climates will be no greater than for pupils who are average in achievement in dominative climates.
3 The amount of achievement for pupils above average in achievement in integrative climates will be no greater than for pupils above average in achievement in dominative climates.
4 The amount of achievement for pupils below average in achievement in integrative climates will be no greater than for pupils who are below average in achievement in dominative climates.

The total sum of squares was divided into a sum of squares for levels, a sum of squares for cells, an interaction sum of squares (treatment by levels), and a within-groups sum of squares.

The results of the analysis of variance are presented in Table 3.

The significance of the difference in main effects was tested by the F ratio using .05 level of confidence. An F ratio of 1.55 was obtained. The probability that an F value of 1.55 or less would occur by chance was more than .05 resulting in the acceptance of the null hypothesis of no difference between the two groups in mathematics achievement. Since the mean square for interaction was less than the within-subgroups mean square (error term), no further F ratios were computed.

On the basis of the obtained data, the null hypothesis was accepted for all of the achievement levels. It was concluded from the results of the obtained data that classroom climate, within the specified limits of this study, did not significantly affect the learning of mathematics by third-grade elementary-school children.

Discussion

The climates in the two groups were evaluated either by a direct rating of the teachers' behaviors or by the behaviors of the pupils, which were assumed to be greatly influenced by the teachers' behaviors. The two groups were differentiated in the amount of enjoyment present in the classroom, the emotions of the teachers, whether the teachers ordered or suggested, and the physical tension of the group. The integrative classrooms were characterized by much enjoyment, the teachers were warm and sympathetic, they avoided coercion, and the pupils were confiding and intimate with each other and with the teacher. In the integrative group, two teachers were rated as warm and sympathetic, seven teachers were rated as friendly and reserved, with depth of contact, and two teachers were rated as straining to keep from expressing irritability. In the dominative group, four of the teachers were rated as being openly hostile and sarcastic toward the pupils in their classes, and seven were rated as being observably irritable in dealing with pupils. It would seem that under these varying conditions the motivations for pupils to

learn would differ. It is possible that pupils in dominative classroom climates are motivated, at least in part, to learn through fear of punishment (teacher aggression).

Several studies [2, 6] indicate that there is little difference in achievement between pupils who are taught under varying methods. However, using the concept of democracy as a frame of reference, the question is raised as to which method is the more supportive of educational philosophy.

Summary

The effects of classroom "climate" on the learning of mathematics by third-grade children were tested using an analysis of variance technique. Two groups were selected using the Wrightstone Teacher-Pupil Rapport Scale. The groups were further divided according to level of achievement. There were high, average, and low achieving levels for each group. The pupils were matched by group and level on initial mathematics achievement, Otis Quick-Scoring I.Q.'s, and chronological age. The schools from which the classes were selected were matched on socioeconomic level, average school achievement, and average school Otis I.Q.

The results indicated that there were no significant differences in mathematics for either group or at any level. It was concluded that classroom climate, as measured by this study, did not have a differential effect on the learning of mathematics by third-grade pupils.

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The Painted Cube—

A wooden cube is painted black on all faces. It is then cut into 27 equal smaller cubes. How many of the smaller cubes are found to be painted on three faces, two faces, one face, and no face?

(*Answer on p.* 380.)

First graders use numbers in opening their school day

Editor's Note

Thanks to Helen Rulla, a first grade teacher at the Madrona School, Seattle, Washington, and to her principal, Olaf Kvamme, for submitting the transcription of the tape recording from which these descriptions were drawn. Miss Rulla explained:

"The recorded experiences were a culmination of many steps taken day by day, week after week. The progress was frequently the result of a child's observation such as, 'We could put two half-dollars together,' etc. Sometimes the next step was deliberately planned by the teacher. It frequently was planned to challenge the more able learner but became common knowledge by repetition." (E. D.)

Taking attendance

TEACHER. Is everyone here at Table Three this morning? Landruff?

LANDRUFF. Yes, everybody's here.

TEACHER. Table Four, is everyone here at your table? Debra?

DEBRA. No, Shirley is not here.

TEACHER. Table Five, Betsy?

BETSY. Lee, Richard, and Ray are not here.

TEACHER. Table One. Gary?

GARY. Yes, everyone is here.

TEACHER. Table Two. Bruce?

BRUCE. No, Sherman's not here.

TEACHER. I'll read the names, and you count to yourselves. See how many children are not here.

TEACHER. Lee, Richard, Shirley, Sherman, Ray. Steven, how many were there?

STEVEN. Five.

TEACHER. How many counted five? That's right. How much is thirty-one take away five? Roland?

ROLAND. Twenty-six.

TEACHER. Let's see if Roland is right. Take away one, it's . . .

CHILDREN. Thirty.

TEACHER. Take away two, it's . . .

CHILDREN. Twenty-nine.

TEACHER. Take away three, it's . . .

CHILDREN. Twenty-eight.

TEACHER. Take away four . . .

CHILDREN. Twenty-seven.

TEACHER. Take away five . . .

CHILDREN. Twenty-six.

TEACHER. We have . . . (*children complete*).

CHILDREN. Twenty-six . . .

TEACHER. Here today. Let's count the children here. (*Teacher points and children count.*)

CHILDREN. One, two, three (*children count to twenty-six, but some, not paying attention, continue to count*) twenty-seven, twenty-eight.

TEACHER. See what happens when we're not watching when we count? We just go on, and our counting isn't right. Now, twenty-six children . . . (*children complete*).

CHILDREN. Are here.

TEACHER. Five children . . .

CHILDREN. Are not here.

TEACHER. We had to take five away.

Buying lunch

TEACHER. Is anyone buying lunch at Table Three? (*No one has brought money.*) Table Four?

TEACHER (*Child is having trouble counting her money*). Let's help Daunetta. She had one dime.

CHILDREN (*counting*). Ten.

TEACHER. Another dime . . .

CHILDREN. Twenty.

TEACHER. A nickel . . .

CHILDREN. Twenty-five.

TEACHER. And another nickel . . .

CHILDREN. Thirty.

TEACHER. Let's count it again (*children count with her*). One dime, ten; another dime, twenty; a nickel, twenty-five; and another nickel, thirty.

CHILDREN (*to Daunetta*). You forgot to put up a marker. Debra, put up a marker.

TEACHER. Debra, if you're buying lunch but you don't have your money, you should come up and tell me so I can write your name down. Otherwise, I'll have four names and there will be five markers, and then we will wonder where the mistake is.

(*Children bring money and count it when handing it to the teacher.*)

MARSHA. One dollar.

CATHY. Twenty-five, thirty.

JIMMY. Ten, twenty, thirty.

TEACHER. Table Five, anyone buying lunch?

JOHN. Twenty-five, twenty-six, twenty-seven, twenty-eight, twenty-nine, thirty.

JULIA. Twenty-five, thirty-five.

TEACHER. Table One?

GARY. Ten, twenty, thirty.

ALAN. Twenty-five, thirty.

ROGER. Twenty-five, thirty, forty.

TEACHER. Table Two?

CECILIA. Twenty-five, thirty.

TEACHER. We forgot to keep track as we went along. How many children were buying at each table? These were the children at Table Four: Marsha, Cathy, Jimmy, Daunetta, and Debra. How many was that? Landruff?

LANDRUFF. Five.

TEACHER. Table Five has Julia and John. How many was that? (*Some say three, others two.*) Five and two are how many? Ila?

ILA. Six.

TEACHER. Five and one are six. Five and two? (*No answer*). Five and one more makes . . .

CHILDREN (*completing*). Six.

TEACHER. One more makes . . .

CHILDREN (*completing*). Seven.

TEACHER. Five and two are seven. Table one was Gary, Alan, and Roger. Larry?

LARRY. Three.

TEACHER. Seven and three are how many?

ALVA. Ten.

TEACHER. We have one ten. Table Two, Meg?

MEG. One.

TEACHER. Ten and one more are how many? Matey?

MATEY. Eleven.

TEACHER. Look at the markers. Who can make the markers show the tens? Ila?

ILA (*counts as she picks up each marker*). One, two, three (*to ten*).

TEACHER. What do the markers tell us, Ila?

ILA. One ten. One ten and one more.

TEACHER. Can you write that number?

ILA. Yes (*writes*).

TEACHER. Now what does the number you wrote say?

ILA. Eleven.

TEACHER. It says eleven. One ten and one more make . . .

CHILDREN (*completing*). Eleven.

TEACHER Is anyone at Table Three buying milk? Landruff?

LANDRUFF. One.

TEACHER. One person is buying milk. Table Four, anybody buying milk? Jimmy?

JIMMY. No.

TEACHER. How much are one and no more? Marsha?

MARSHA. One.

TEACHER. It doesn't change, does it?

If you don't have any more, it doesn't change.

A CHILD. If you had one more it would be two.

TEACHER. Yes, but it doesn't change because we had no more. Table Five, anybody buying milk? So, it's still the same, isn't it? One and zero . . .

CHILDREN (*laughing*). Makes one.

TEACHER. Table One, anybody buying milk? Daunetta?

DAUNETTA. One more.

TEACHER. One and one are . . .

CHILDREN. Two.

TEACHER. Table Two?

CHILDREN. One, two, three . . . three!

TEACHER. Three and two are how many? Three and two? Betsy?

BETSY. Five.

TEACHER. How much are two and three? Gary?

GARY. Five.

TEACHER. Just the same, isn't it? Two in this hand and three over here?

CHILDREN. Five.

TEACHER. If I had three in the first hand and two in the other?

CHILDREN. Five.

TEACHER. Just the same . . . three and two are five, two and three are five. (*Children count money as they give it to the teacher.*)

LARRY. Five cents.

PRUDENCE. One, two, three . . . three cents.

BRUCE. Five cents.

TEACHER. How many children are buying milk today? Jackson?

JACKSON. Five.

TEACHER. Can you write the number for us? (*Jackson writes.*) Now look at the markers. Do we have another ten?

CHILDREN. No.

TEACHER. Jimmy?

JIMMY. Sixteen.

TEACHER. Do we have another ten?

JIMMY. No.

TEACHER. How many more do we have? Lonnie?

LONNIE. Five.

TEACHER. Five for milk but the other one makes it . . .

CHILDREN. Six.

TEACHER. So, now we have ten and . . .

CHILDREN. Six.

TEACHER. That's what Jimmy was telling us. Altogether there are . . .

CHILDREN. Sixteen.

TEACHER. Write the number for us, Jimmy. (*Jimmy writes.*) Jimmy, tell us how much sixteen is.

JIMMY. One ten and six more.

TEACHER. Is it more than eleven? How much is eleven?

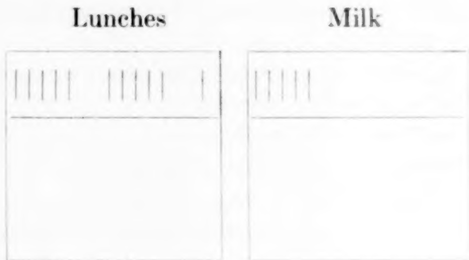
CHILDREN. One ten and one more.

TEACHER. How much is five?

JIMMY. Just five.

TEACHER. Yes, it doesn't even make a ten, only half of a ten.

CHILDREN. It takes another five to make ten. (*The chart shows the tally with markers as arranged by children to indicate the number of lunches and milk orders.*)



TEACHER. Now, let's see how much money we have this morning. (*Children complete blanks on chalkboard as lesson progresses.*)

We have ? dollars. ____	1	$1.00
We have ? half dollars. ____	0	$0.—
We have ? quarters. ____	6	$1.50
We have ? dimes. ____	12	$1.20
We have ? nickels. ____	8	$ .40
We have ? pennies. ____	8	$ .08

How much money is this? Julia? (*Holds up a dollar bill.*)

JULIA. One dollar.

TEACHER. On the board it asks how many dollars. Who can write the number on that first line? We have how many dollars? Gary? (*Gary writes one.*) At the end of the line it asks how much money that is. Who can write how much money it is? Who can write one dollar? Marsha? (*Marsha writes one dollar in figures.*) Who can read the next line? Sharon?

SHARON. We have how many half dollars?

TEACHER. Roger, what number would you write to show how many half dollars?

ROGER. Zero.

TEACHER. Look at the next line, what do we want to know? Roland?

ROLAND. How many quarters?

TEACHER (*shows quarters*). What number would you write, Roland? (*He writes six.*)

TEACHER. Six quarters. Who can tell us how much money six quarters make?

PRUDENCE. One dollar and fifty cents.

TEACHER. Can you write that on the line? (*Prudence writes.*) Lonnie, can you read the next line for us?

LONNIE. We have how many dimes?

TEACHER. Now, we'll have to watch again and see how many dimes there are. Everybody ready? Pierre, how many dimes did you count?

PIERRE. Twelve.

TEACHER. Can you write that on the board for us? Where it says we have how many dimes? Who can say that in another way? Pierre said we have twelve dimes. Can you tell me in another way? John?

JOHN. One dozen.

TEACHER. One dozen dimes. Let's see how much one dozen dimes is. One dime is . . . (*Children count by tens to 100.*)

CHILDREN. One dollar.

TEACHER. One dollar ten, one dollar twenty. So, we have one dollar and twenty cents.

TEACHER. Who can write that on the board? Twelve dimes are one dollar and twenty cents. Sharon? How many dollars can you see? Larry?

LARRY. Three.

TEACHER. Three dollars. We have two quarters. How much is that? Betsy?

BETSY. Fifty cents.

TEACHER. Fifty cents and a dime make . . .

CHILDREN. Sixty.

TEACHER. And one other dime . . .

CHILDREN. Seventy.

TEACHER. Now, who can read the next line? Prudence?

PRUDENCE. We have how many nickels?

TEACHER. Everybody watch to see how many nickels there are. How many nickels did you count? Cecilia?

CECILIA. Eight.

TEACHER. How many counted eight? Write the number for us Cecilia. While she's doing that, let's count and see how much money eight nickels are. One nickel is . . . (*children count to forty by fives*). Now let's put them with the other money. We had seventy cents . . . a nickel makes . . . (*children count to 100 by fives*). One hundred cents is one dollar. Another dollar, how how many dollars are there?

CHILDREN. Four.

TEACHER. Four, and we have two nickels.

CHILDREN. Five, ten.

TEACHER. So, there are . . .

CHILDREN. Four dollars and ten cents.

TEACHER. We forgot to write down how much eight nickels make. Do you remember? Marsha? How much was it?

MARSHA. Forty cents.

TEACHER. Yes, you write it for us. Look at the pennies. How many pennies in this pile? Sharon?

SHARON. Eight.

TEACHER. Eight pennies. Eight pennies are how much money?

CHILDREN. Eight cents.

TEACHER. Will you write eight cents, Bruce? Now, let's see how much money we have altogether. This is . . .

CHILDREN. One dollar.

TEACHER. Four quarters are . . .

CHILDREN. One dollar.

TEACHER. Another dollar. That makes how much?

CHILDREN. Two dollars.

TEACHER. Another dollar . . .

CHILDREN. Three dollars.

TEACHER. And another . . .

CHILDREN. Four dollars.

TEACHER. Four dollars and two nickels . . .

CHILDREN. Five, ten.

TEACHER. And eight pennies . . .

CHILDREN. Eleven, twelve . . . eighteen.

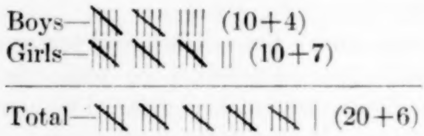
TEACHER. So we have how much money altogether?

CHILDREN. Four dollars and eighteen cents.

Extensions to other grade levels

These necessary activities, like taking attendance and buying lunch, continue throughout the grades. The number experiences which accompany them may increase in difficulty as children gain more understanding and greater skill in mathematics. You might like to try these methods of taking attendance if they seem appropriate for your grade level.

1. Tally by fives. Translate tally marks into ten and so many more.



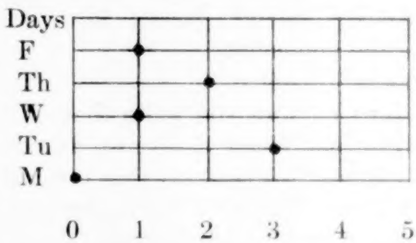
2. Count by twos, threes, fours. Children may arrange themselves in groups of three or four to facilitate counting in multiples, as shown in the charts in the next column.

   This scheme can provide readiness for multiplication or practice in multiplication facts with remainders.

	Groups of 3	Left over	Total
Boys	5	2	15+2 or 17
Girls	4	2	12+2 or 14
Children	10	1	30+1 or 31

	Groups of 4	Left over	Total
Boys	4	1	16+1 =17
Girls	3	2	12+2 =14
Children	7	3	28+3 =31

3. Make and compare graphs of absences for two weeks or for two months.



Number of children absent

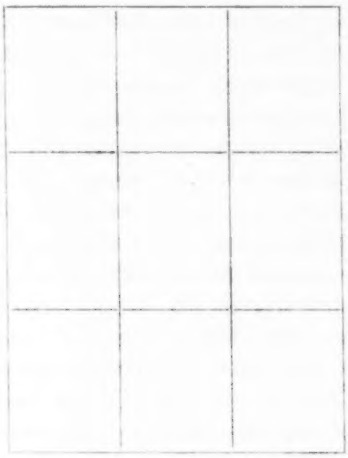
4. Give the job of taking attendance to a pair of children and allow them to decide on methods which correspond with their level of maturity and grasp of mathematics.

For each pupil with a spark of genius, there are ten with ignition trouble.
—*Mississippi Educational Advance*

A device for practice with common denominators and addition of unlike fractions

ROBERT C. HAMMOND *Baldwinsville, New York*

The square in the accompanying figure and the numbers 1–9 have often been used to form a "magic square" in which the numbers in all rows, columns, and diagonals can be arranged to add to 15. This magic square can be adapted so that it becomes a device or game which provides practice in forming common denominators and adding unlike fractions (fractions having different denominators).



Two pupils or contestants play this game. The numbers 0, 2, 4, 6, and 8 are placed over 15. The fractions thus formed are reduced if possible and assigned to one pupil. Likewise, the numbers 1, 3, 5, 7, and 9 are placed over 15. Again the fractions formed are reduced if possible and assigned to the other pupil. The two resulting sets of fractions are (1) $\frac{0}{15}$, $\frac{2}{15}$, $\frac{4}{15}$, $\frac{2}{5}$, $\frac{8}{15}$ and (2) $\frac{1}{15}$, $\frac{1}{5}$, $\frac{1}{3}$, $\frac{7}{15}$, $\frac{3}{5}$.

To play the game, the two pupils alternately each place one of their fractions in one of the small squares. Any fraction may be used only once per game. The winner is the player who places a third fraction in a row, column, or diagonal so that the fractions in the row, column, or diagonal add to $\frac{15}{15}$ or 1.

Winners may go on to play other winners so that a class champion emerges. The writer has had most success, however, when players are approximately equal in ability. Also, a player should win at least two games before being called a winner and moving on to play someone else. It will be found that many games end in a tie.

The principles or understandings involved in this device are of course those involved in forming common denominators and adding unlike fractions. To have the most practice with the principles involved, pupils should have the opportunity to play with both sets of fractions.

This device is appropriate at the junior high school or the upper elementary-school levels. Rapid learners should be encouraged to develop games of this type using other fractions. Furthermore, this same magic square structure can be adapted for other purposes. In the lower grades, for example, practice in adding whole numbers can be provided. In this case, one pupil would be assigned the numbers 1, 3, 5, 7, and 9; another pupil the numbers 0, 2, 4, 6, and 8. Now the winner would be the player who places a third whole number in a row, column, or diagonal so that the numbers in the row, column, or diagonal give the sum of 15.

Geometry for primary grades

NEWTON S. HAWLEY
Project Director, Stanford University, Stanford, California

Editor's Note

The project on Geometry for Primary Grades deserves to be known by all persons concerned with programs of elementary-school mathematics instruction. The following information regarding this project is taken, with minor editorial changes, from a report prepared by Professor Hawley for distribution to interested persons. (*J.F.W.*)

This program for teaching geometry in the elementary grades had its beginning in the spring of 1958 when an informal experiment was conducted in the teaching of geometry in a first grade class. At that time the class received a geometry lesson lasting about twenty minutes each school day—usually in the early mornings. This first experiment lasted about two months. The material presented was based on Euclid's *Elements* but was presented, of course, in a considerably modified manner.

At that time the students learned how to construct an equilateral triangle, then they proceeded to the bisection of angles and line segments. Perpendiculars to a line were constructed, then perpendiculars through a point on the line, through a point off the line, and through an end-point of a line segment. These basic constructions led to the discovery of more complex constructions. The high level of comprehension of the students was most encouraging.

At the beginning of the initial experiment there had been no plan to continue this work, but the success was so pronounced and the children were so delighted, that the idea evolved of extending the work to a wider domain. Worksheets were prepared for distribution to a limited number of classes the following semester. With these worksheets, the geometry lessons no longer required a teacher with highly specialized training in this subject. A further advantage of such printed matter was the fact that the children were required to *read* the material themselves. In this way the geometry program strongly supplemented the reading program. Further, the *type* of reading was new for this age group and served as an introduction to written material of a type that usually has been deferred until much higher grade levels. The necessity for the students to have a rudimentary skill in reading meant, of course, that the geometry program now could not be introduced until the upper half of the first grade or perhaps until the second grade.

In the summer of 1959, this geometry project received support from the National Science Foundation to conduct a large-scale experiment involving the teaching of geometry in the primary grades. The Carnegie Corporation of New York provided support in the production of a bound workbook and of supplementary materials necessary to conduct such an experiment. During the summer of 1959, a project was organized which involved four entire school districts and also a

number of individual schools outside these districts. These additional schools included private schools and parochial schools. More than 3,000 children in over 100 classes were involved in that year.

During that first year, extra copies of the workbooks had been printed so that they could be made available at cost to persons who might be interested. However, so many orders and requests for information about the workbooks were received as news of the project spread that the project was unable to handle the matter efficiently. For that reason it was decided to allow a textbook publisher to print and sell a revised version of the original workbooks.

In the second year of the geometry program the project grew somewhat. Further work was done with the students who already had completed a year's work in geometry; also, new students entered the program. More than 5,000 students came under the direct supervision of the project, with hundreds of other students involved in schools which are conducting a geometry program at their own expense with practically no assistance from the project. This refers to schools in the San Francisco peninsula area. There are also many schools throughout the country and in other parts of the world which are using this material and conducting their own programs. The project has little contact with them, except for occasional correspondence. Such outside information is always welcome, even though the project currently is not organized in such a manner that any appreciable assistance can be offered to these outside schools.

Aims and objectives

This geometry material is designed for introduction at the second- or third-grade level. It should be emphasized that the material is intended for use by *all* students rather than by only the "gifted" child. The treatment is not axiomatic, hence the work bears little resemblance to the usual high school course.

Geometry has an immediate appeal to young students because of the geometrical figures which they can construct themselves with instruments they enjoy using. It became apparent early in the experiment that the students liked the material in itself, and that there was no need whatever to "sugar-coat" it. The fact that students at this grade level actually develop a genuine intellectual interest in material of significant content is important in itself.

It is also desirable to introduce earlier a branch of mathematics which is in many ways more typical of mathematics as a whole than is arithmetic. *However, geometry should not be considered as a replacement for arithmetic. The techniques learned in arithmetic are essential in themselves.* With geometry, there is a need for ingenuity on the part of the student if he is to solve some of the problems. Also, it becomes apparent that there may be more than one correct way to solve certain problems.

Geometry was long been recognized for its value as an introduction to analytical and creative thinking, and this is certainly a very strong reason to introduce it at the primary-grade level. With a thorough grasp of geometry, the student will be in a better position to understand and to analyze the physical world. Moreover, the important concept of *precision* is introduced in a most effective manner.

Another important aspect of the geometry program is the intimate connection with the reading program. The student must *read* the instructions in the workbook in order to be able to carry out the steps necessary to complete the construction required. The teacher has an immediate test for comprehension, since the student cannot proceed with the construction unless he understands the instructions. The material which the student has to read increases in complexity throughout the course. Hence, this material also serves as an introduction to reading that special type of writing which requires analytical thinking on the part of the reader if he is

to understand exactly what is being said. Too often, students are not required to face this problem.

There are students with exceptional intellectual potential whose abilities often go unnoticed when judged by their performance in the standard school curriculum. However, these students often are stimulated and challenged by geometry. In the first year of the project such children were, in fact, discovered. If such students are allowed to languish throughout their formative years until near maturity, their potential may disappear or remain forever undeveloped. The world cannot afford to lose these exceptional minds.

Materials and equipment needed

The following pupil workbooks, with accompanying teacher manuals, now are available:* *Geometry for Primary Grades, Book One; Geometry for Primary Grades, Book Two; Teachers Manual for Book One; Teachers Manual for Book Two.*

In addition, one needs straightedges and compasses. The straightedge for geometry preferably has no numbers or calibrations on it, since the straightedge is used only to draw straight lines and never to measure with. However, if such a straightedge is not available, a plastic ruler will serve.

The compass should be a bow compass with five-inch arms and *screw adjustment.* The ordinary friction-setting compass of the type usually found in the variety store is not satisfactory as the setting tends to slip during use. The compass should be capable of drawing circles of diameter up to eight inches.

A more complete discussion of the equipment and its proper use, along with various suggestions for teaching the course, can be found in the teachers manuals mentioned above.

* Pupil workbooks and teacher manuals currently are available from Holden-Day, Inc., Publishers, 728 Montgomery Street, San Francisco 11, California.

Can you do it?

You have a quarter to spend. You purchase some drawing supplies, spending 25 cents for 25 articles. You buy four kinds of articles: paper at two sheets for a cent, pens at a cent apiece, pencils at two for a nickel, and erasers at a nickel each. How many of each kind did you take?

Answer: Fourteen sheets of paper, 8 pens, 2 pencils, and 1 eraser.

Books and materials

Extending Mathematical Ideas, Ginn Arithmetic Enrichment Program for Grade 6, John L. Marks, James R. Smart, and Irene Sauble. Boston: Ginn and Co., 1961. Paper, 96 pp., $0.64.

Number Principles and Patterns, Ginn Arithmetic Enrichment Program for Junior High School, Allene Archer. Boston: Ginn and Co., 1961. Paper, 68 pp., $0.76.

Both of the books reviewed are of the paper-bound workbook type. They are clearly written, contain excellent mathematical terminology, and use literal factors to a great degree throughout the text. While explanations are very good, the books do need additional interpretation and explanation by the teacher due to the type of material they contain. The content of the books is exceptional and should be included in any arithmetic program that seeks additional material for the average and better students.

Most of the material in these two workbooks is not in any one textbook series, and they contain much that is not in any series at all. Much of the new mathematics that is currently in demand by most teachers is a part of the content.

Some of the topics in the sixth-grade book are: base five and eight arithmetic, clock arithmetic, scratch methods of addition and subtraction, rules for compensation in arithmetic, negative numbers, probability, geometry, set theory, primes and factors, exponents and squares, number sequences, variables, and the associative and commutative principles. It also contains a section on the metric system which has much material on linear units, some on liquid measure, but omits metric units of weight, area, volume, and temperature, which I feel should have been included in a book of this type.

The mechanical make-up of the sixth-grade book is not the best; some pages are printed in two vertical columns, and others are printed in the conventional manner. This becomes confusing where one topic extends over two or more pages and the columns vary from one page to the next.

Some of the topics covered in the junior high book are: the associative, distributive, and commutative principles; the property of closure; number patterns in sequences; tests for divisibility; ten-tens charts for the basic operations; nomographs; squares and square roots; primes and factors; and positive rational numbers.

The mechanical make-up of this book is good, the terminology is exceptional, and the content is presented in excellent order. I feel that a section on set language should have been presented as a follow-up to the introduction given in the sixth-grade book. This omission, however, in no way detracts from the excellent material that is presented.

Based on the contents alone, I feel that both of these books belong in every upper-grade classroom as part of the arithmetic enrichment program.

THEODORE S. KOLESNIK
Assistant Principal
Cleveland School
Skokie, Illinois

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Registrations at NCTM meetings

Perhaps the most significant development in the attendance at NCTM conventions during the 1960–61 year was the unusually large attendance at the Thirty-ninth Annual Meeting, with an official registration of 3,232. This figure was nearly 1,200 greater than the registration at the largest previous annual meeting.

Below are official registration reports for all of the NCTM meetings held during the 1960–61 year. The total number of persons present at the site of each meeting was somewhat greater than the number reported here for two reasons. (1) The official registration report does not include members of families or friends who did not come primarily to attend the meeting. (2) There are always a few convention participants who fail to take the time to register formally.

Registrations at the Twentieth Summer Meeting

The National Council of Teachers of Mathematics, Salt Lake City, Utah, August 21–24, 1960

Arizona	12	New Hampshire	2
Arkansas	1	New Jersey	10
California	76	New Mexico	2
Colorado	24	New York	14
Connecticut	2	North Carolina	2
District of Columbia	8	Ohio	17
Florida	2	Oklahoma	3
Georgia	3	Oregon	17
Idaho	7	Pennsylvania	5
Illinois	56	Rhode Island	2
Indiana	16	South Carolina	1
Iowa	10	South Dakota	1
Kansas	17	Tennessee	2
Louisiana	7	Texas	10
Maine	2	Utah	178
Maryland	4	Virginia	2
Massachusetts	10	Washington	16
Michigan	14	Wisconsin	13
Minnesota	7	Wyoming	9
Missouri	10	Canada	9
Montana	21	Foreign	7
Nebraska	7		
Nevada	1	Total	639

Registrations at the Nineteenth Christmas Meeting

The National Council of Teachers of Mathematics, Tempe, Arizona, December 28–30, 1960

Arizona	260
Arkansas	1
California	111
Colorado	18
Connecticut	4
District of Columbia	5
Florida	2
Georgia	1
Illinois	36
Indiana	9
Iowa	7
Kansas	3
Louisiana	2
Maryland	2
Massachusetts	4
Michigan	5
Minnesota	6
Missouri	13
Nebraska	3
Nevada	3
New Jersey	3
New Mexico	12
New York	6
North Dakota	4
Ohio	5
Oklahoma	3
Oregon	7
Pennsylvania	6
Rhode Island	1
Tennessee	5
Texas	21
Utah	1
Virginia	1
Washington	1
Wisconsin	5
Canada	4
Foreign	1
TOTAL	581

Registrations at the Thirty-ninth Annual Meeting

The National Council of Teachers of Mathematics, Chicago, Illinois, April 5–8, 1961

Alabama	9
Arizona	3
Arkansas	12
California	46
Colorado	12
Connecticut	26
Delaware	4
District of Columbia	23
Florida	29
Georgia	12
Idaho	2
Illinois	1,319
Indiana	211
Iowa	63
Kansas	32
Kentucky	12
Louisiana	13
Maine	1
Maryland	29
Massachusetts	45
Michigan	197
Minnesota	83
Mississippi	6
Missouri	165
Nebraska	15
New Hampshire	4
New Jersey	50
New Mexico	2
New York	108
North Carolina	7
North Dakota	10
Ohio	183
Oklahoma	22
Oregon	3
Pennsylvania	90
Rhode Island	2
South Carolina	3
South Dakota	4
Tennessee	40
Texas	27
Utah	2
Vermont	1
Virginia	26
Washington	3
West Virginia	7
Wisconsin	228
Wyoming	1
Canada	38
Foreign	2
TOTAL	3,232

Registrations at the Joint Meeting with the National Education Association

The National Council of Teachers of Mathematics, Atlantic City, New Jersey, June 28, 1961

Alabama	3
Alaska	1
Arizona	1
Arkansas	1
California	10
Colorado	1
Connecticut	5
Delaware	10
District of Columbia	2
Florida	2
Georgia	3
Hawaii	2
Idaho	2
Illinois	16
Indiana	5
Iowa	12
Kansas	4
Kentucky	3
Louisiana	2
Maryland	14
Massachusetts	8
Michigan	10
Minnesota	1
Mississippi	6
Missouri	4
Nebraska	2
New Jersey	78
New York	24
North Carolina	3
Ohio	20
Oklahoma	7
Oregon	3
Pennsylvania	54
Rhode Island	1
South Carolina	3
South Dakota	2
Tennessee	4
Texas	5
Utah	2
Vermont	2
Virginia	11
Washington	4
West Virginia	6
Wisconsin	1
Wyoming	1
Puerto Rico	1
TOTAL	361

"Almost a hundred years ago mathematicians began to be bothered by the fact that real numbers developed without their supervision, like plants in an untended garden. Fortunately, however, they were able to prove that the end product (the real numbers) of this unsupervised growth is good; in other words, they proved that there is a logical plan, which, had it been followed, would have yielded the real number system as we know it today."

—From *Basic Concepts In Modern Mathematics* by John E. Hafstrom

Answer (The Painted Cube): Only one of the 27 small cubes is unpainted; 8 are painted on three faces, 12 on two faces, 6 on one face. These numbers correspond to the number of corners, edges, and faces of the large cube.